Partitions with Distinct and Repeated Parts

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This article starts with a brief introduction to partitions. Then we turn to partition-generating functions that produce a q-polynomial in which the coefficient of q^n is the number of partitions for the integer n. This relationship between these polynomial coefficients and partitions is fascinating in the same way that the q-factorial and the q-binomial coefficient were related to the number of inversions (see the articles in the Inversions topic).

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapter 3 of *An Introduction to q-analysis* by Warren P. Johnson.

1 Partitions

How many ways can an integer can be represented as a sum of positive integers? Each way is called a "partition," which consists summands called "parts." Table 1 shows the partitions for integers up to 6.

Table 1: Partition Examples

n	Partitions
2	(2), (1, 1)
3	(3), (2, 1), (1, 1, 1)
4	(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)
5	(5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1)
6	(6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1), (2 1 1 1 1), (1 1 1 1 1 1)

There two partitions of 2, three partitions of 3, five partitions of 4, seven partitions of 5, and eleven partitions of 6.

In this article we are interested in partition-generating functions that count the number of partitions whose parts are distinct, repeated, odd, or even. For example, for n = 5 the parts in the first three partitions are distinct, 1 is repeated in the fourth, sixth, and seventh partitions, and 2 is repeated in the fifth partition.

2 Generating Functions Overview

We demonstrate seven generating functions that count the number of partitions whose parts are distinct, repeated, odd, and/or even.

The generating functions are defined in terms of infinite q-shifted factorials of the form $(q;q)_\infty$ and $\frac{1}{(q;q)_\infty}$. The partition-generating functions are shown in Table 2.

Table 2: Partition Generating Functions

#	Generating Function	Type of Partition
1	$(-q;q)_\infty$	distinct parts
2	$(-q;q^2)_\infty$	distinct odd parts
3	$(-q^2;q^2)_\infty$	distinct even parts
4	$\frac{1}{(q;q)_\infty}$	all parts, possibly repeated
5	$\frac{1}{(q;q^2)_\infty}$	all odd parts, possibly repeated
6	$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}$	distinct (#1) = odd (#5) (Euler's theorem)
7	$(-q^2;q^2)_\infty \times \frac{1}{(q;q^2)_\infty} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}$	odd repeated and even distinct (#3 x #5)
8	$\frac{(q^4;q^4)_\infty}{(q;q)_\infty}$	no part used more than three times
9	$\frac{(-q^3;q^3)_\infty}{(q^2;q^2)_\infty}$	no singleton, i.e., no part exactly once

3 Partitions with Distinct Parts

I will use n = 5 and n = 6 as examples for the first three generating functions in Table 2.

3.1 Distinct Parts

The generating functions that count the number of distinct parts is $(-q;q)_\infty$. The low-order terms of the generated polynomial are 5*q**7 + 4*q**6 + 3*q**5 + 2*q**4 + 2*q**3 + q**2 + q + 1.

Looking at the list of partitions in Table 1, we see that for n = 5 there are three partitions with distinct parts (5), (4 1), (3 2), and hence the coefficient of the q**5 term is 3.

For n = 6, there are four partitions with distinct parts (6), (5 1), (4 2), (3 2 1), resulting in a coefficient of 4 on the q**6 term.

3.2 Distinct Odd Parts

The generating functions that count the number of distinct odd parts is $(-q;q^2)_\infty$. The low-order terms of the generated polynomial are q**7 + q**6 + q**5 + q**4 + q**3 + q + 1.

There is one partition of 5 with distinct odd parts (5) and one partition of 6 with distinct odd parts (5 1), resulting in coefficients of 1 on both the q**6 and q**5 terms.

3.3 Distinct Even Parts

The generating functions that count the number of distinct even parts is $(-q^2;q^2)_\infty$. The low-order terms of the generated polynomial are 2*q**8 + 2*q**6 + q**4 + q**2 + 1.

There are no partitions of 5 with distinct even parts, and thus there is no q**5 term. There are two partitions of 6 with distinct even parts (6), (4 2), which is indicated by the coefficient of 2 on the q**6 term.

4 Partitions with Repeated Parts

4.1 Repeated Parts (All Parts)

$\frac{1}{(q;q)_\infty}$ is the generating function for partitions in which some parts may be repeated, but are not necessarily repeated. Because there are no restrictions on the parts, this function simply counts all the parts. The lower order terms of the polynomial are 15*q**7 + 11*q**6 + 7*q**5 + 5*q**4 + 3*q**3 + 2*q**2 + q + 1.

The coefficients agree with the partitions shown in Table 1, i.e., there two partitions of 2, three partitions of 3, five partitions of 4, seven partitions of 5, and eleven partitions of 6.

4.2 Odd Repeated

$\frac{1}{(q;q^2)_\infty}$ is the generating function for partitions with all odd parts, possibly repeated. The low-order terms of the generated polynomial are 5*q**7 + 4*q**6 + 3*q**5 + 2*q**4 + 2*q**3 + q**2 + q + 1.

Now for n = 5 there are three partitions with possibly repeated odd parts: (5), (3 1 1), (1 1 1 1 1); hence the coefficient of 3 on the q**5 term.

5 Combinations

5.1 Distinct Equals Repeated Odd

Euler demonstrated that the generating function for distinct parts produces the same number of partitions as the generating function for possibly repeated odd parts. The generating function identity is $(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}$. The low-order terms of the generated polynomials are:

Distinct Parts : 6*q**8 + 5*q**7 + 4*q**6 + 3*q**5 + 2*q**4 + 2*q**3 + q**2 + q + 1

Repeated Odd Parts: 6*q**8 + 5*q**7 + 4*q**6 + 3*q**5 + 2*q**4 + 2*q**3 + q**2 + q + 1

5.2 Odd Repeated and Even Distinct

$\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}$ is the generating function for partitions with possibly repeated odd parts and distinct even parts. Note that this is a product of the generating function for repeated odd $\frac{1}{(q;q^2)_\infty}$ and distinct even $(-q^2;q^2)_\infty$, demonstrating that the product of two generating functions “ands” their effects.

The low-order terms of the generated polynomial are 12*q**7 + 9*q**6 + 6*q**5 + 4*q**4 + 3*q**3 + 2*q**2 + q + 1.

For example, for n = 5 observe that there are six partitions with possibly repeated odd parts but distinct even parts: (5), (4 1), (3 2), (3 1 1), (2 1 1 1), (1 1 1 1 1); hence the coefficient of the q**5 term is 6.

6 Restrictions

6.1 No Part More than Three Times

The function $\frac{(q^4;q^4)_\infty}{(q;q)_\infty}$ generates partitions in which no part is used more than three times. The low-order terms of the generated polynomial are 12*q**7 + 9*q**6 + 6*q**5 + 4*q**4 + 3*q**3 + 2*q**2 + q + 1. Compare this to the polynomial generated in Section 5.2 for partitions with possibly repeated odd parts and distinct even parts. They are the same. In other words it can be proven that $\frac{(q^4;q^4)_\infty}{(q;q)_\infty} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}$.

For n = 5, there are six partitions in which no part is used more than three times: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), which is reflected by the coefficient of 6 on the q**5 term.

6.2 Not Singleton

$\frac{(-q^3;q^3)_\infty}{(q^2;q^2)_\infty}$ is the generating function for partitions in which no part appears exactly once, and also the generating function for partitions in which <u>all</u> parts are congruent to 0, 2, 3, or 4 mod 6.

The low-order terms of the generated polynomial are 2*q**7 + 4*q**6 + q**5 + 2*q**4 + q**3 + q**2 + 1.

For n=5, recall that there are a total of seven partitions: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1). There is one partition in which no part appears exactly once: (1 1 1 1 1), and one partition in which all parts are congruent to 0, 2, 3, or 4 mod 6: (3 2). Hence, the coefficient of 1 on the q**5 term.

For n = 6, recall that there are 11 partitions: (6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1), (2 1 1 1 1), (1 1 1 1 1 1). There are four partitions in which no part appears exactly once: (3 3), (2 2 2), (2 2 1 1), (1 1 1 1 1), and four partitions in which all parts are congruent to 0, 2, 3, or 4 mod 6: (6), (4 2), (3 3), (2 2 2). Hence, the coefficient of 4 on the q**6 term.

