Divisors

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In this article, instead of counting the number of partitions for an integer *n*, we count or sum the integers that divide *n*. Counts and sums are used in some interesting identities and theorems. First we use divisor counts starting with Lambert's theorem, which leads to theorems by Uchimura and Eisenstein. Then we use divisor sums for Euler's amazing divisor sum theorem and finish with Euler's phi function identity.

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapter 4 of *An Introduction to q-analysis* by Warren P. Johnson.

1 Divisor Counts and Sums

Table 1 shows the divisor counts and sums for the divisors of *n*.

Table 1: Divisor Counts and Sums

n	Divisors	Count	Sum
4	[1, 2, 4]	3	7
5	[1, 5]	2	6
6	[1, 2, 3, 6]	4	12
7	[1, 7]	2	8
8	[1, 2, 4, 8]	4	15
9	[1, 3, 9]	3	13
10	[1, 2, 5, 10]	4	18
11	[1, 11]	2	12
12	[1, 2, 3, 4, 6, 12]	6	28
13	[1, 13]	2	14
14	[1, 2, 7, 14]	4	24
15	[1, 3, 5, 15]	4	24
16	[1, 2, 4, 8, 16]	5	31
17	[1, 17]	2	18
18	[1, 2, 3, 6, 9, 18]	6	39
19	[1, 19]	2	20
20	[1, 2, 4, 5, 10, 20]	6	42

2 Lambert's Theorem

Let $d(n)$ denote the number of divisors for n. Then Lambert's theorem is $\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} = \sum_{n=1}^{\infty} d(n) q^n$.

This equality can be demonstrated by choosing an arbitrary value for |q| < 1. For q = 0.5 the equality improves with increasing *inf* as shown in Table 2. Since there are no components of the form $\sum_{k=1}^{\infty} \frac{1}{(q;q)_k}$, it is computationally feasible to use large values to approximate infinity.

Table 2: Lambert's Theorem			
inf	$\sum_{k=1}^{\infty} \frac{q^k}{1-q^k}$	$\sum_{n=1}^{\infty} d(n) q^n$	Gap
7	1.59098822324629	1.56250000000000	2.849%
10	1.60474075478420	1.59960937500000	0.513%
15	1.60663411601725	1.60644531250000	0.0189%
20	1.60669324506545	1.60668563842773	0.000761%
30	1.60669515055265	1.60669514164329	0.00000089%

3 Uchimura's Theorem

Lambert's theorem together with Euler's curious little identity in the article Connection between Inversions and q-Binomial Coefficients are used to derive Uchimura's theorem

$$(q;q)_\infty \sum_{k=1}^{\infty} \frac{kq^k}{(q;q)_k} = \sum_{n=1}^{\infty} d(n) q^n .$$

For q = 0.5 the equality improves with increasing *inf* as shown in the table below. Since there are components of the form $\sum_{k=1}^{\infty} \frac{1}{(q;q)_k}$, it is not computationally feasible to use large values to approximate infinity.

Table 3: Uchimura's Theorem			
inf	$(q;q)_\infty \sum_{k=1}^{\infty} \frac{kq^k}{(q;q)_k}$	$\sum_{n=1}^{\infty} d(n) q^n$	Gap
6	1.37761423	1.5	8.16%
7	1.47542045	1.5625	5.57%
8	1.53291335	1.578125	2.86%

4 Eisenstein's Theorem

Lambert's theorem, together with a variant of Euler's curious little identity in the article Connection between Inversions and q-Binomial Coefficients are used to derive Eisenstein's theorem $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{kq^{\binom{k+1}{2}}}{(q;q)_k}=(q;q)_\infty\sum_{n=1}^{\infty}d(n)q^n$. For q = 0.5 the equality improves with increasing *inf* as shown in the table below.

Table 4: Eisenstein's Theorem			
inf	$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{kq^{\binom{k+1}{2}}}{(q;q)_k}$	$(q;q)_\infty\sum_{n=1}^{\infty}d(n)q^n$	Gap
6	0.46040021	0.440021753	4.43%
7	0.4622061	0.454775087	1.61%
8	0.46310369	0.457528608	1.20%

5 Euler's Divisor Sum Theorem

This is a very satisfying theorem because it circles back and connects with Euler's pentagonal number theorem. Letting $\sigma(n)$ denote the sum of the divisors as shown in Table 1, Lambert's theorem is used to derive Euler's divisor sum theorem:

$$\sigma(n)=\sigma(n-1)+\sigma(n-2)-\sigma(n-5)-\sigma(n-7)+\sigma(n-12)+\sigma(n-15)\ldots$$.

Note that each term is the divisor count of n minus the exponents from the pentagonal number theorem, every second one of which is a pentagonal. When n is a pentagonal number, instead of the last term being $\sigma(0)$, it is n.

For example, for n = 56, we have $\sigma(56)$ =120 = 72 +120 -72 -57 +84 +42 -54 -72 +32 +31 -6.

6 Euler's Phi Identity

6.1 Euler's Phi Function

Now, instead of the number of divisors of n, we consider the integers that are not divisors of n. The Euler phi function $\varphi(n)$ counts the number of positive integers $\leq n$ that are relatively prime (co-prime) to n. An integer m is co-prime to n if the gcd(m,n) = 1. The co-primes and Euler phi function are listed up to 20 in Table 5.

Table 5: Euler's Phi Function

n	**integers co-prime to *n***	$\varphi(n)$
4	[1, 3]	2
5	[1, 2, 3, 4]	4
6	[1, 5]	2
7	[1, 2, 3, 4, 5, 6]	6
8	[1, 3, 5, 7]	4
9	[1, 2, 4, 5, 7, 8]	6
10	[1, 3, 7, 9]	4
11	[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]	10
12	[1, 5, 7, 11]	4
13	[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]	12
14	[1, 3, 5, 9, 11, 13]	6
15	[1, 2, 4, 7, 8, 11, 13, 14]	8
16	[1, 3, 5, 7, 9, 11, 13, 15]	8
17	[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]	16
18	[1, 5, 7, 11, 13, 17]	6
19	[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]	18
20	[1, 3, 7, 9, 11, 13, 17, 19]	8

6.2 Euler's Phi Identity

Denoting the Euler phi function $\varphi(n)$ and using Lambert's theorem we can derive the identity

$$\frac{q}{(1-q)^2} = \sum_{n=1}^{\infty} \frac{\varphi(n) q^n}{1-q^n}.$$

For q = 0.5 the equality improves with increasing *inf* as shown in the table below. As was the case for Lambert's theorem, it is computationally feasible to use large values to approximate infinity.

Table 6: Euler's Phi Identity

inf	$\frac{q}{(1-q)^2}$	$\sum_{n=1}^{\infty} \frac{\varphi(n) q^n}{1-q^n}$	Gap
7	2	1.913159242	8.684%
10	2	1.987831294	1.217%
15	2	1.999434617	0.0565%
20	2	1.999980129	0.001987%
30	2	1.99999997	0.00000297%

