Ferrers Diagram

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This article explains the Ferrers diagram and its components, the Durfee square and the Franklin triangle. These diagram components are then used to prove Jacobi's Durfee square identity.

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapter 4 of *An Introduction to q-analysis* by Warren P. Johnson.

1 Ferrers Diagrams

Ferrers diagrams were developed by British mathematician Norman Ferrers in the 1800s. In Section 4.2 of *An Introduction to q-analysis*, Johnson uses three partitions: 8 5 4 3 3 1, 9 6 5 4 3 1, and 8 6 4 3 2 1. The Ferrers diagrams for these three partitions are shown in Table 1.

Table 1: Basic Ferrers Diagrams

8 5 4 3 3 1	**9 6 5 4 3 1**	**8 6 4 3 2 1**
* *	* *	* *

A Ferrers diagram can be divided into four components:

- A <u>square</u> in the upper left, called the Durfee square
- A <u>triangle</u> to the right of the Durfee square, called the Franklin triangle
- The remaining components on the right
- The remaining components on the left

A Python program for Ferrers diagrams calculates the size of the Durfee square by starting at the upper left corner and checking for squares. Then it uses the size of the Durfee square to change the Ferrers diagram to show the Durfee square (*), the Franklin triangle (F), and the right (R) and lower (L) components. The completed diagrams are illustrated in Table 2.

Table 2: Complete Ferrers Diagrams		
8 5 4 3 3 1	**9 6 5 4 3 1**	**8 6 4 3 2 1**
* * * F F F R R * * * F F * * * F L L L L L L L	* * * * F F F R R * * * * F F * * * * F * * * * L L L L	* * * F F F R R * * * F F R * * * F L L L L L L

Note that the first and third diagrams have Franklin triangles with the same dimension as the Durfee square (k = 3). In the second diagram, the Durfee square has dimension k = 4, but the Franklin triangle has dimension k – 1 = 3.

2 Jacobi's Durfee Square Identity

The Ferrers diagram is used as the basis for the proof of Jacobi's Durfee square identity $\frac{1}{(xq;q)_\infty}$ $= 1+\sum_{k=1}^{\infty}\frac{x^k q^{k^2}}{(q;q)_k(xq;q)_k}$. We know from the second Euler identity in the article Euler's Partition Identities that the left side of Jacobi's Durfee square identity $\frac{1}{(xq;q)_\infty}$ is the generating function for partitions with exactly *k* parts.

Now we have two identities for this generating function: Euler's identity $\frac{1}{(xq;q)_\infty} = \sum_{k=0}^{\infty}\frac{x^k q^k}{(q;q)_k}$ and Jacobi's identity $\frac{1}{(xq;q)_\infty} = 1+\sum_{k=1}^{\infty}\frac{x^k q^{k^2}}{(q;q)_k(xq;q)_k}$

2.1 Ferrers Diagram Components

Now, the right side $1+\sum_{k=1}^{\infty}\frac{x^k q^{k^2}}{(q;q)_k(xq;q)_k}$ of the identity is interesting from the geometric perspective since it is derived from the Durfee square, and the right and lower components of the Ferrers diagram. For this generating function, the Durfee square contributes $x^k q^{k^2}$, the right component contributes $\frac{1}{(q;q)_k}$, and the lower component contributes $\frac{1}{(xq;q)_k}$.

For example, for k = 5, the individual components of the Ferrers diagram are:

Durfee square: $x^5 q^{5^2}$ = q**25*x**5. The exponent of q is the number of dots in the Durfee square.

Right component: $\frac{1}{(q;q)_5}$ produces a polynomial in q of order 399. Of interest to us are the low-order terms 7*q**5 + 5*q**4 + 3*q**3 + 2*q**2 + q + 1, specifically the term for q**5. The coefficient (7) of this term is the number of partitions of 5.

Lower component: $\frac{1}{(xq;q)_5}$ produces a very long polynomial with highest order term q**399*x**114. The coefficient of x**5 is a polynomial with low-order terms 2*q**7 + q**6 + q**5. The coefficients of these terms are the number of partitions with 5 parts. For example, for 5 the single partition is (1 1 1 1 1) and for 6 it is (2 1 1 1 1). For 7, there are two partitions with five parts: (3, 1, 1, 1, 1), (2, 2, 1, 1, 1).

2.2 Demonstrating the Identity

A Python program demonstrates Jacobi's Durfee square identity $\frac{1}{(xq;q)_\infty}$ =

$$1+\sum_{k=1}^{\infty}\frac{x^k q^{k^2}}{(q;q)_k (xq;q)_k}.$$

A comparison of the low-order terms of the q-polynomial coefficients of the *x* terms in Table 3 demonstrate the identity.

Table 3: Demonstrating the Identity

x**k	$\frac{1}{(xq;q)_\infty}$	$1+\sum_{k=1}^{\infty}\frac{x^k q^{k^2}}{(q;q)_k (xq;q)_k}$
x**4	3*q**7 + 2*q**6 + q**5 + q**4	3*q**7 + 2*q**6 + q**5 + q**4
x**3	4*q**7 + 3*q**6 + 2*q**5 + q**4 + q**3	4*q**7 + 3*q**6 + 2*q**5 + q**4 + q**3
x**2	3*q**7 + 3*q**6 + 2*q**5 + 2*q**4 + q**3 + q**2	3*q**7 + 3*q**6 + 2*q**5 + 2*q**4 + q**3 + q**2

