# Pentagonal Numbers

by Alan Mehlenbacher

This article starts with polygonal numbers in general and then explains Euler's discovery that $(-q;q)_\infty$ is a polynomial for which the pentagonal numbers are represented by every second exponent. This is a very interesting relationship between pentagonal numbers and q-series.

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapter 4 of *An Introduction to q-analysis* by Warren P. Johnson.

## 1 Polygonal Numbers

Polygonal numbers are a series of numbers for $n = 1,\ldots,\infty$ that are based on polygons. The triangular number series begins with 1, 3, 6, 10, 15, 21; the square numbers with 1, 4, 9, 16, 25, 36; the pentagonal numbers with 1, 5, 12, 22, 35, 51; and the hexagonal numbers with 1, 6, 15, 28, 45, 66. These are generated using a simple formula, but they also have a geometric interpretation.

For n = 1, we start with one dot. For n = 2 we add dots to form the shape of the polygon, and for subsequent n we extend each side by one dot. This process is illustrated in the figures below.

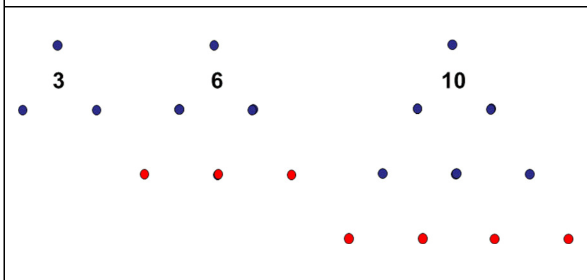
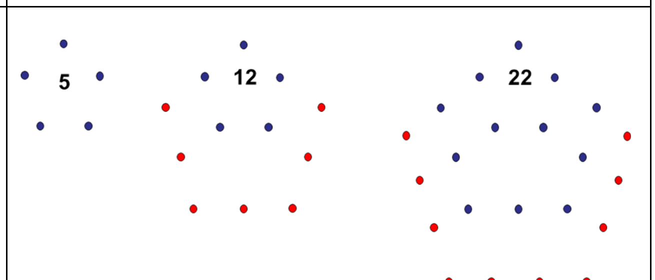
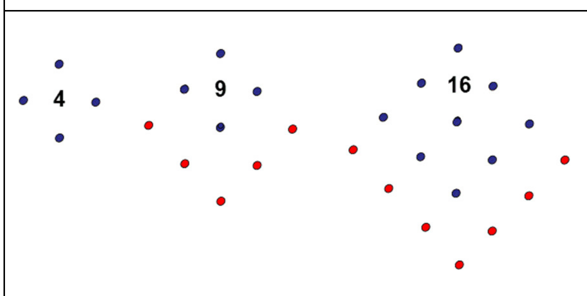
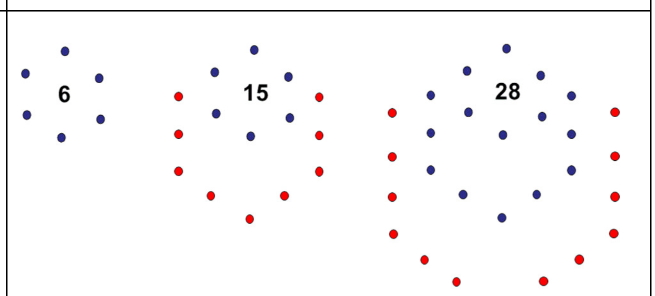
| Triangular Numbers | Pentagonal Numbers |
| --- | --- |
|  |  |
| **Square Numbers** | **Hexagonal Numbers** |
|  |  |

Table 1 shows the polygonal numbers for n = 1 to 20.

| **Table 1: Polygonal Numbers** | |
|---|---|
| Triangular | 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210 |
| Square | 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400 |
| Pentagonal | 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, 210, 247, 287, 330, 376, 425, 477, 532, 590 |
| Hexagonal | 1, 6, 15, 28, 45, 66, 91, 120, 153, 190, 231, 276, 325, 378, 435, 496, 561, 630, 703, 780 |
| Heptagonal | 1, 7, 18, 34, 55, 81, 112, 148, 189, 235, 286, 342, 403, 469, 540, 616, 697, 783, 874, 970 |
| Octagonal | 1, 8, 21, 40, 65, 96, 133, 176, 225, 280, 341, 408, 481, 560, 645, 736, 833, 936, 1045, 1160 |

# 2 Euler's Pentagonal Number Theorem

The highlight of this topic is Euler's pentagonal number theorem, which is a product-sum identity: $(q;q)_\infty = 1+\sum_{k=1}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}}(1+q^k)$.

## 2.1 $(-q;q)_\infty$

Euler first observed that $(-q;q)_\infty$ is a polynomial for which the pentagonal numbers are represented by every <u>second</u> exponent.

Approximating infinity with 20 for $(-q;q)_\infty$ produces pentagonal numbers up to only q**15. Using 50 for infinity results in accurate exponents up to only q**51. Euler solved this problem by deriving a sum that is identical to the product.

## 2.2 $1+\sum_{k=1}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}}(1+q^k)$

Euler derived the summation expression on the right side of this equation because it provides an accurate result for any large value that is used for infinity. For example, using 20 for the summation quickly and accurately produces the first 20 pentagonal numbers:

q**610 + q****590** - q**551 - q****532** + q**495 + q****477** - q**442 - q****425** + q**392 + q****376** - q**345 - q****330** + q**301 + q****287** - q**260 - q****247** + q**222 + q****210** - q**187 - q****176** + q**155 + q****145** - q**126 - q****117** + q**100 + q****92** - q**77 - q****70** + q**57 + q****51** - q**40 - q****35** + q**26 + q****22** - q**15 - q****12** + q**7 + q****5** - q**2 – q****1** + 1

This convincingly demonstrates the value of Euler's theorem, and generally the value of the summation term in a product-sum identity.

.