# Euler's Partition Identities

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Recall that partition-generating functions produce a $q$-polynomial in which the coefficient of $q^n$ is the number of partitions for the integer $n$. The first article of the Partitions topic demonstrated nine partition-generating functions for partitions with distinct, repeated, odd, and/or even parts and two generating functions that applied single restrictions related to the number of parts. The second article demonstrated five partition-generating functions that applied two restrictions to the partitions. This article features three of Euler's partition-generating function identities and concludes with the valuable identity for the $q$-binomial series.

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapter 3 of *An Introduction to q-analysis* by Warren P. Johnson and Chapter 1 of *Number theory in the Spirit of Ramanujan* by Bruce C. Berndt.

## 1 Generating Functions Overview

Table 1 lists the partition-generating function identities that are demonstrated in this article. Note that there are two generating functions in each identity, a product on the left and a sum on the right.

**Table 1: Partition Generating Function Identities**

| # | Generating Function Identity | Type of Partition |
|---|---|---|
| 1 | $\frac{1}{(x;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k}$ | Euler identity: at most k parts |
| 2 | $\frac{1}{(xq;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k q^k}{(q;q)_k}$ | Euler identity: exactly k parts |
| 3 | $(-x;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{(q;q)_k}$ | Euler identity: at most k distinct parts |
| 4 | $\frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{k=0}^{\infty} \frac{(a;q)_k x^k}{(q;q)_k}$ | $q$-binomial series |

 

Table 2 shows the partitions for n = 2 to 6 to be used to demonstrate the theorems.

| **Table 2: Partition Examples** | |
|---|---|
| **n** | **Partitions** |
| 2 | (2), (1, 1) |
| 3 | (3), (2, 1), (1, 1, 1) |
| 4 | (4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1) |
| 5 | (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1) |
| 6 | (6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1), (2 1 1 1 1), (1 1 1 1 1 1) |

# 2 The Euler Partition-Generating Function Identities

The three Euler identities introduce *x* into the partition-generating function, which then produces coefficients of the $x^k$ that are q-polynomials. These polynomials are generating functions for partitions with at most k parts, exactly k parts, or at most k distinct parts.

## 2.1 At Most k Parts

Euler provides an identity for the generating function $\frac{1}{(x;q)_\infty}$, which generates polynomials for partitions with at most k parts. The identity is $\frac{1}{(x;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k}$ . The polynomials are accurate up to the order ∞-1, For example, using 7 to approximate infinity, we obtain polynomials that are accurate up to the sixth order term.

Using $x^4$ for the example, both sides produce a coefficient of $x^4$ with low-order terms 9*q**6 + 6*q**5 + 5*q**4 + 3*q**3 + 2*q**2 + q + 1.

This indicates that there are nine partitions of 6 and six partitions of 5 with <u>at most</u> 4 parts. Referring to the partitions of 6 and 5 shown above, we can confirm that this is correct, namely (6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1) for 6 and (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1) for 5.

 

## 2.2 Exactly k Parts

Euler provides an identity for the generating function $\frac{1}{(xq;q)_\infty}$, which generates polynomials for partitions with exactly k parts. The identity is $\frac{1}{(xq;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k q^k}{(q;q)_k}$ . Again using 7 to approximate infinity, both sides produce a coefficient of $x^3$ with low-order terms 3*q**6 + 2*q**5 + q**4 + q**3. This indicates that there are three partitions of 6 and two partitions of 5 with exactly 3 parts. Referring to the partitions of 6 and 5 shown above, we can confirm that this is correct, namely (4 1 1), (3 2 1), (2 2 2) for 6 and (3 1 1), (2 2 1) for 5.

## 2.3 At Most k Distinct Parts

This partition theorem of Euler provides an identity for the generating function $(-x;q)_\infty$, which generates polynomials for partitions with at most k distinct parts. The identity is $(-x;q)_\infty$ = $\sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{(q;q)_k}$ . Using 7 to approximate infinity, both sides produce a coefficient of $x^3$ with low-order terms 3*q**6 + 2*q**5 + q**4 + q**3. This indicates that there are three partitions of 6 and two partitions of 5 with at most 3 distinct parts. Referring to the partitions of 6 and 5 shown above, we can confirm that this is correct. For 6 we have (5 1), (4 2), (3 2 1) and for 5 we have (4 1), (3 2).

 

## 3 q-Binomial Series

Euler developed slightly different versions of two of the identities above. In addition to the first identity $\frac{1}{(x;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k}$, he showed that $\frac{1}{(bx;q)_\infty} = \sum_{k=0}^{\infty} \frac{b^k x^k}{(q;q)_k}$. In addition to the third identity $(-x;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{(q;q)_k}$, Euler showed that $(ax;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} (-a)^k x^k}{(q;q)_k}$.

These identities were combined by Cauchy and Jacobi into a q-binomial series (with b = 1) that is written very elegantly as $\frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{k=0}^{\infty} \frac{(a;q)_k x^k}{(q;q)_k}$. This is a very important identity, and is also a theorem in Chapter 1 of Berndt's *Number theory in the Spirit of Ramanujan*. A sample of results demonstrates the equality:

The low-order terms of the coefficient of x**3 for the left-side are 7*q**7 + 7*q**6 + 5*q**5 + 4*q**4 + 3*q**3 + 2*q**2 + q + 1 and for the right side they are 7*q**7 + 7*q**6 + 5*q**5 + 4*q**4 + 3*q**3 + 2*q**2 + q + 1.

The low-order terms of the coefficient of x**2 for the left-side are 3*q**7 + 4*q**6 + 3*q**5 + 3*q**4 + 2*q**3 + 2*q**2 + q + 1 and for the right side 3*q**7 + 4*q**6 + 3*q**5 + 3*q**4 + 2*q**3 + 2*q**2 + q + 1.