${}_1\psi_1$ *q*-Hypergeometric Series

This article introduces a special type of q-hypergeometric series, the ${}_1\psi_1$ function. In general, the "psi" q-hypergeometric series is distinguished from the "phi" q-hypergeometric series in the previous topic (see the article "q-Hypergeometric Series") by having the same number of terms in the numerator as in the denominator; in other words, the notation is ${}_r\psi_r$ instead of ${}_{r+1}\phi_r$.

We start with a nicely balanced ${}_1\psi_1$ sum-product identity by Ramanujan, from which we derive two versions of Cauchy's ${}_1\psi_1$ identity. In the second article for this topic (Sums of Squares), we use the second version of the Cauchy ${}_1\psi_1$ identity to derive a polynomial for which the coefficient of q^m is the number of ways of writing m as a sum of <u>two</u> squares. We use the first version of the Cauchy ${}_1\psi_1$ identity to derive a polynomial for which the coefficient of q^m is the number of ways of writing m as a sum of <u>four</u> squares.

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapters 6 and 7 of *An Introduction to q-analysis* by Warren P. Johnson.

1 Introducing ${}_1\psi_1$

The simple q-hypergeometric series in my article "q-Hypergeometric Series" was ${}_2\phi_1$ = $\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}x^n$. The function we are using in this topic is ${}_1\psi_1$, where ${}_1\psi_1$ = $\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}x^n$.

This demonstrates three characteristics of a psi q-hypergeometric series that distinguishes it from a phi q-hypergeometric series:

- There is no $(q;q)_n$ term in the denominator.
- The number of numerator terms is the same as the number of denominator terms.
- The summation is over both negative and positive integers.

2 Ramanujan's ${}_1\psi_1$ Identity

Ramanujan discovered a sum-product identity for ${}_1\psi_1$ in terms of eight q-shifted factorials:

$${}_1\psi_1 = \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty \left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty \left(\frac{b}{ax};q\right)_\infty (b;q)_\infty \left(\frac{q}{a};q\right)_\infty}.$$

Johnson explains at least four proofs of Ramanujan's identity in Section 6.1 of *An Introduction to q-analysis.*

For convergence, we require $|q|<1$ and $\left|\frac{b}{a}\right|<|x|<1$. In addition to these constraints, we must avoid an infinite number of parameter values that make a q-shifted factorial in the denominator zero. For example, we must avoid $a = q$, $a=q^2$, … , $b = ax$, $b = 1$, $b = 1/q$, $b = 1/q^2$, … .

To study convergence, I used a Python function over a range of feasible values for a, c, x, and q.

The results in Table 1 show increasing accuracy as the value of *inf* increases. The column labels are values for q and the row labels are values for a within values for x. The values for b are not shown, but were calculated using $a*x-0.01$. We see that the slowest convergence occurs when there is a large value of q in combination with small values of the other parameters.

Table 1: Ramanujan $_1\psi_1$ Identity

Percent Difference between Sum and Product

(rows: a within x; columns: q)

inf = 20

Average of PerDiff	Column Labels						
Row Labels	**0.13**	**0.26**	**0.39**	**0.52**	**0.65**	**0.78**	**0.91**
0.14							
0.15	0.00	0.00	0.00	0.00	0.00	0.00	1,230.14
0.30	0.00	0.00	0.00	0.02	0.01	0.68	7.72
0.45	0.01	0.06	0.06	0.06	0.04	3.06	0.69
0.60	0.04	0.01	0.03	0.03	0.13	10.17	0.99
0.75	0.47	0.38	0.46	0.50	0.84	2.49	1.00
0.90	0.99	0.40	0.53	0.68	0.28	1.32	1.00
0.28							
0.15	0.00	0.00	0.00	0.01	0.00	0.16	34.06
0.30	0.03	0.01	0.02	0.02	0.09	6.96	0.03
0.45	0.95	0.38	0.50	0.63	0.24	1.12	1.00
0.60	0.30	0.41	0.21	0.35	0.94	1.54	1.00
0.75	0.34	0.47	0.37	0.48	0.45	1.24	1.00
0.90	0.36	1.00	0.42	0.75	0.79	0.83	0.99
0.42							
0.15	0.01	0.04	0.04	0.04	0.01	0.77	7.69
0.30	0.93	0.37	0.48	0.60	0.23	0.98	0.99
0.45	0.32	0.42	0.32	0.40	0.08	0.43	1.00
0.60	0.34	0.95	0.40	0.70	0.72	0.66	0.99
0.75	0.36	0.28	0.46	0.31	0.36	0.51	1.08
0.90	0.64	0.62	1.12	0.69	0.84	0.10	1.05
0.56							
0.15	0.03	0.00	0.02	0.01	0.06	3.18	3.28
0.30	0.29	0.38	0.20	0.32	0.79	1.21	0.92
0.45	0.33	0.91	0.38	0.66	0.66	0.55	0.99
0.60	0.67	0.64	0.77	0.67	0.73	0.92	0.98
0.75	0.60	0.58	0.42	0.67	3.41	0.80	0.79
0.90	0.56	0.55	0.52	1.11	0.59	0.48	0.68
0.70							
0.15	0.41	0.32	0.38	0.40	0.60	2.07	0.87
0.30	0.30	0.41	0.32	0.40	0.34	0.77	1.04
0.45	0.32	0.26	0.41	0.26	0.30	0.32	1.23
0.60	0.58	0.57	0.40	0.65	3.22	0.72	1.09
0.75	0.53	0.52	0.50	1.11	0.57	0.59	1.49
0.90	0.72	0.71	0.71	0.67	1.11	0.46	1.50
0.84							
0.15	0.73	0.30	0.38	0.46	0.17	0.58	0.84
0.30	0.28	0.73	0.31	0.52	0.50	0.34	0.86
0.45	0.58	0.56	0.97	0.60	0.70	0.00	1.28
0.60	0.50	0.49	0.46	0.94	0.49	0.34	1.54
0.75	0.68	0.68	0.67	0.64	1.02	0.41	0.23
0.90	0.84	0.84	0.83	0.83	0.81	1.06	0.40
Grand Total	**0.42**	**0.42**	**0.39**	**0.48**	**0.61**	**1.33**	**36.46**

inf = 100

Average of PerDiff	Column Labels						
Row Labels	**0.13**	**0.26**	**0.39**	**0.52**	**0.65**	**0.78**	**0.91**
0.14							
0.15	0.00	0.00	0.00	0.00	0.00	0.00	41.24
0.30	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.45	0.00	0.00	0.00	0.00	0.00	0.00	0.03
0.60	0.00	0.00	0.00	0.00	0.00	0.00	0.05
0.75	0.01	0.01	0.01	0.01	0.02	0.08	1.17
0.90	0.02	0.01	0.01	0.01	0.01	0.03	3.64
0.28							
0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.01
0.30	0.00	0.00	0.00	0.00	0.00	0.00	0.01
0.45	0.02	0.01	0.01	0.01	0.00	0.02	2.20
0.60	0.01	0.01	0.00	0.01	0.02	0.03	0.88
0.75	0.01	0.01	0.01	0.01	0.01	0.02	0.01
0.90	0.01	0.02	0.01	0.02	0.02	0.01	0.07
0.42							
0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.30	0.02	0.01	0.01	0.01	0.00	0.02	1.46
0.45	0.01	0.01	0.01	0.01	0.00	0.01	0.09
0.60	0.01	0.02	0.01	0.01	0.01	0.01	0.05
0.75	0.01	0.01	0.01	0.01	0.01	0.01	0.17
0.90	0.12	0.11	0.20	0.12	0.15	0.00	0.52
0.56							
0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.30	0.01	0.01	0.00	0.01	0.02	0.02	0.39
0.45	0.01	0.02	0.01	0.01	0.01	0.01	0.04
0.60	0.16	0.15	0.18	0.16	0.17	0.21	0.30
0.75	0.09	0.09	0.06	0.10	0.50	0.11	0.54
0.90	0.06	0.06	0.05	0.12	0.06	0.05	0.35
0.70							
0.15	0.01	0.01	0.01	0.01	0.01	0.04	0.22
0.30	0.01	0.01	0.01	0.01	0.01	0.02	0.01
0.45	0.01	0.01	0.01	0.01	0.01	0.01	0.07
0.60	0.08	0.08	0.06	0.09	0.47	0.10	0.42
0.75	0.05	0.05	0.05	0.11	0.06	0.06	0.16
0.90	0.20	0.20	0.20	0.19	0.31	0.13	0.27
0.84							
0.15	0.02	0.01	0.01	0.01	0.00	0.01	0.36
0.30	0.01	0.02	0.01	0.01	0.01	0.01	0.01
0.45	0.10	0.10	0.18	0.11	0.13	0.00	0.25
0.60	0.05	0.05	0.05	0.10	0.05	0.04	0.17
0.75	0.19	0.19	0.19	0.18	0.29	0.11	0.21
0.90	0.44	0.44	0.44	0.44	0.43	0.56	0.81
Grand Total	**0.05**	**0.05**	**0.05**	**0.05**	**0.08**	**0.05**	**1.56**

3 Cauchy's First ${}_1\psi_1$ Identity

About 80 years before Ramanujan, as explained in Section 7.1 of Johnson's book, Cauchy developed an identity that is a special case of Ramanujan's identity when b is replaced with aq.

3.1 Summation Derivation

Starting with Ramanujan's summation $\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n$, set $b = aq$ so that the denominator is now $(aq;q)_n$. Expanding this using the definition of the q-shifted factorial (see the article "Curious and Complicated q-Binomial Theorems"), we have

$(aq;q)_n = \frac{1-a}{1-a}(1-aq)(1-aq^2)\cdots(1-aq^{n-1})(1-aq^n) = \frac{(a;q)_n}{1-a}(1-aq^n)$. Substituting into the summation term yields $(1-a)\sum_{n=-\infty}^{\infty} \frac{1}{(1-aq^n)} x^n$.

3.2 Product Derivation

Replacing b with aq in Ramanujan's product $\frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty \left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty \left(\frac{b}{ax};q\right)_\infty (b;q)_\infty \left(\frac{q}{a};q\right)_\infty}$ results in the product term $\frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty (q;q)_\infty}{(x;q)_\infty \left(\frac{q}{x};q\right)_\infty (aq;q)_\infty \left(\frac{q}{a};q\right)_\infty}$. Dividing both sides by $(1-a)$ changes the $(aq;q)_\infty$ term to $(1-a)\,(aq;q)_\infty = (a;q)_\infty$, and we have the Cauchy ${}_1\psi_1$ identity: $\sum_{n=-\infty}^{\infty} \frac{1}{(1-aq^n)} x^n =$ $\frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty^2}{(x;q)_\infty \left(\frac{q}{x};q\right)_\infty (a;q)_\infty \left(\frac{q}{a};q\right)_\infty}$. We use this first Cauchy ${}_1\psi_1$ identity for sums of four squares in the article "Sums of Squares."

3.3 Convergence

The constraints for convergence are $|q| < |x| < 1$ and $|q| < |a| < 1$. To avoid a zero denominator, we must also avoid values $a = q$, $1/q$, $1/q^2$, …, q^2, q^3…, and $x = q$, $1/q$, $1/q^2$, …, q^2, q^3… . To achieve these constraints, I use ranges for x and a but calculate $q = x \times a$.

To study convergence, I implemented a Python function over a range of feasible values for a, x, and q.

The results in Table 2 show rapid convergence as the value of *inf* increases. The row labels are values for *x,* and the columns labels are values for *a*. The calculated values for *q* range from 0.02 to 0.16, and the results are averaged over these values.

Table 2: Cauchy's First $_1\psi_1$ Identity

Percent Difference between Sum and Product

(rows: *x*; columns: a, averaged over q = {0.02, …, 0.16}

inf = 20

Average of Diff	Column Labels					
Row Labels	**0.14**	**0.28**	**0.42**	**0.56**	**0.70**	**0.84**	**Grand Total**
0.14	0.0000	0.0000	0.0000	0.0000	0.0043	0.3273	0.0553
0.28	0.0000	0.0000	0.0000	0.0000	0.0043	0.3273	0.0553
0.42	0.0000	0.0000	0.0000	0.0000	0.0019	0.1553	0.0262
0.56	0.0000	0.0000	0.0000	0.0000	0.0023	0.1870	0.0316
0.70	0.0019	0.0019	0.0019	0.0018	0.0008	0.2072	0.0359
0.84	0.1606	0.1606	0.1606	0.1606	0.1577	0.0642	0.1440
Grand Total	**0.0271**	**0.0271**	**0.0271**	**0.0271**	**0.0286**	**0.2114**	**0.0580**

inf = 50

Average of Diff	Column Labels					
Row Labels	**0.14**	**0.28**	**0.42**	**0.56**	**0.70**	**0.84**	**Grand Total**
0.14	0.0000	0.0000	0.0000	0.0000	0.0000	0.0032	0.0005
0.28	0.0000	0.0000	0.0000	0.0000	0.0000	0.0032	0.0005
0.42	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007	0.0001
0.56	0.0000	0.0000	0.0000	0.0000	0.0000	0.0010	0.0002
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0012	0.0002
0.84	0.0009	0.0009	0.0009	0.0009	0.0009	0.0006	0.0008
Grand Total	**0.0001**	**0.0001**	**0.0001**	**0.0001**	**0.0001**	**0.0016**	**0.0004**

4 Cauchy's Second ${}_1\psi_1$ Identity

For this second identity derived by Cauchy, replace q with q^2 and x with xq in the summation and product to obtain $\sum_{n=-\infty}^{\infty} \frac{x^n q^{2n}}{1+q^{2n}} = \frac{(axq;q^2)_\infty \left(\frac{q}{ax};q^2\right)_\infty (q^2;q^2)_\infty^2}{(xq;q^2)_\infty \left(\frac{q}{x};q^2\right)_\infty (a;q^2)_\infty \left(\frac{q^2}{a};q^2\right)_\infty}$.

Then set $a = -1$ to obtain $\sum_{n=-\infty}^{\infty} \frac{x^n q^{2n}}{1+q^{2n}} = \frac{(-xq;q^2)_\infty \left(-\frac{q}{x};q^2\right)_\infty (q^2;q^2)_\infty^2}{(xq;q^2)_\infty \left(\frac{q}{x};q^2\right)_\infty (-1;q^2)_\infty \left(-q^2;q^2\right)_\infty}$.

Note that $(-1;q^2)_\infty = 2(1+q^2)(1+q^4)\cdots = 2\left(-q^2;q^2\right)$, resulting in the second Cauchy ${}_1\psi_1$ identity $\sum_{n=-\infty}^{\infty} \frac{2x^n q^{2n}}{1+q^{2n}} = \frac{(-xq;q^2)_\infty \left(-\frac{q}{x};q^2\right)_\infty (q^2;q^2)_\infty^2}{(xq;q^2)_\infty \left(\frac{q}{x};q^2\right)_\infty \left(-q^2;q^2\right)_\infty^2}$. We use this second Cauchy ${}_1\psi_1$ identity for sums of two squares in the article "Sums of Squares."

Again requiring $|q| < |x| < 1$, or equivalently $|q| < |x| < \frac{1}{q}$, and avoiding problem values for x such as $x = q, q^3, q^5, \ldots, 1/q, 1/q^3, 1/q^5, \ldots$, the identity quickly converges.