Partition Theory

Introduction

How many ways can an integer can be represented as a sum of positive integers? Each way is called a "partition," which consists summands called "parts." For example, there are seven partitions for 5, in which the parts in each partition sum to 5: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1).

In Section I, we start with a brief introduction that calculates all the partitions for an integer *n*.

In the second and third parts, we turn to partition-generating functions that produce a *q*-polynomial in which the coefficient of q^n is the number of partitions for the integer *n*. This relationship between these polynomial coefficients and partitions is surprising in the same way that the q-binomial coefficient was related to the number of inversions for each permutation (see the Inversions topic).

In Section II, we study partition-generating functions that that contain infinite $qq\infty$ functions of the form $(q;q)_\infty$ and $\frac{1}{(q;q)_\infty}$. In the Python code, I approximate $\frac{1}{(q;q)_\infty}$ using geometric series.

In Section III, we study generating functions that contain finite q-shifted factorials of the form $(q;q)_n$ and $\frac{1}{(q;q)_n}$. I approximate $\frac{1}{(q;q)_n}$ using an approximation by Cauchy.

See Chapter 3 of *An Introduction to q-analysis* by Warren P. Johnson for detailed explanations, derivations, and proofs. The purpose of these Python functions is to provide examples that demonstrate these relatively simple but amazing functions.

Section I: Partitions

Before turning to partition-generating functions, I provide code for an introduction to partitions.

PrintAllPartitions(n): Calls **CalculateAllPartitions(n)** and prints the partitions.

CalculateAllPartitions(n): Calculates all partitions of *n*.

For example, there are seven partitions of 5: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1), and there are 11 partitions of 6: (6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1), (2 1 1 1 1), (1 1 1 1 1 1).

Most of the generating functions that we study in Sections I and II are used to count the number of partitions whose parts are distinct, repeated, odd, even, or less than a certain value. For example, the parts in the first three partitions of 5 are distinct, 1 is repeated in the fourth, sixth, and seventh partitions, and 2 is repeated in the fifth partition.

Section II: Generating Functions–Infinite

In this section, I provide Python code for the generating functions that contain infinite $qq\infty$ functions of the form $(q;q)_\infty$ and $\frac{1}{(q;q)_\infty}$. These partition-generating functions are shown in the table below.

#	Generating Function	Type of Partition
1	$(-q;q)_\infty$	distinct parts
2	$(-q;q^2)_\infty$	distinct odd parts
3	$(-q^2;q^2)_\infty$	distinct even parts
4	$\frac{1}{(q;q)_\infty}$	all parts, possibly repeated
5	$\frac{1}{(q;q^2)_\infty}$	all odd parts, possibly repeated
6	$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}$	distinct (#1) = odd (#5) (Euler's theorem)
7	$(-q^2;q^2)_\infty \times \frac{1}{(q;q^2)_\infty} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}$	repeated odd, distinct even (#3 x #5)
8	$(-q^2;q^2)_\infty \times (-q^2;q^2)_\infty \times \frac{1}{(q;q)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}$	no part used more than three times (#3 x #3 x #4)
9	$\binom{i+j}{i}_q$	at most i parts, each part at most j (Cayley's theorem)
10	$\frac{(-q^3;q^3)_\infty}{(q^2;q^2)_\infty}$	no singleton, i.e., no part exactly once

qqninf(A,B,inf): Calculates functions of the form $(A;B)_\infty$, such as $(q;q)_\infty$. Using this simple example, we define $(q;q)_\infty$ as the infinite version of $(q;q)_n$, which was defined in the q-Binomial theorems topic. Thus, $(q;q)_\infty = (1-q)(1-q^2)(1-q^3)(1-q^4)...$ I suggest using 20 for infinity.

qqninfInverse(A,B, inf): Calculates functions of the form $\frac{1}{(A;B)_\infty}$, such as $\frac{1}{(q;q)_\infty}$, using the geometric series.

Using the definition of $(q;q)_\infty$, we have $\frac{1}{(q;q)_\infty} = \frac{1}{(1-q)...}\frac{1}{(1-q^2)}\frac{1}{(1-q^3)}\frac{1}{(1-q^4)}...$. Requiring that $|q|<1$, each one of these terms can be approximated using the geometric series $\frac{1}{1-r}$ = $1+r+r^2+r^3+r^4+...$. Thus, $\frac{1}{(q;q)_\infty}$ = $(1+q+q^2+q^3+q^4+...)$ $(1+q^2+q^4+q^6+q^8...)$ $(1+q^3+q^6+q^9+q^{12}...)$ $(1+q^4+q^8+q^{12}+q^{16}...)$ ...

Because of the geometric series approximation, this inverse requires many more calculations than $(A;B)_\infty$. I suggest using 10 or less (depending on your computer) as good enough, because the intent is to run the functions for small values in order to demonstrate the theorems. The resulting polynomial is valid up to the value you use for approximating infinity.

GeometricSeries(r): Calculates the geometric series for 1/(1-r).

PrintCoeffs(expr,n): Converts a Sympy partition expression to a Sympy polynomial, extracts coefficients, reverses order, and prints first *n* coefficients.

Partitions with Distinct Parts

PartitionsDistinct(inf): Demonstrates the generating functions shown in the table below.

Generating Function	Type of Partition
$(-q;q)_\infty$	partitions with distinct parts
$(-q;q^2)_\infty$	partitions with distinct odd parts
$(-q^2;q^2)_\infty$	partitions with distinct even parts

I will use n = 5 and n = 6 as examples for these three generating functions. There are seven partitions of 5: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1). There are 11 partitions of 6: (6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1), (2 1 1 1 1), (1 1 1 1 1 1).

For distinct parts: The low-order terms of the generated polynomial are 5*q**7 + 4*q**6 + 3*q**5 + 2*q**4 + 2*q**3 + q**2 + q + 1. Looking at the list of partitions, we see that for n = 5 there are three partitions with distinct parts (5), (4 1), (3 2), and hence the coefficient of the q**5 term is 3. For n = 6, there are four partitions with distinct parts (6), (5 1), (4 2), (3 2 1), resulting in a coefficient of 4 on the q**6 term.

For distinct odd parts: The low-order terms of the generated polynomial are q**7 + q**6 + q**5 + q**4 + q**3 + q + 1. There is one partition of 5 with distinct odd parts (5) and one partition of 6 with distinct odd parts (5 1), resulting in coefficients of 1 on both the q**6 and q**5 terms.

For distinct even parts: The low-order terms of the generated polynomial are 2*q**8 + 2*q**6 + q**4 + q**2 + 1. There are no partitions of 5 with distinct even parts, and thus there is no q**5 term. There are two partitions of 6 with distinct even parts (6), (4 2), which is indicated by the coefficient of 2 on the q**6 term.

PartitionsRepeated(inf): Demonstrates that $\frac{1}{(q;q)_\infty}$ is the generating function for partitions in which some parts may be repeated. Because there are no restrictions on the parts, this function produces all the partitions. For example, for n = 5 there are seven partitions with possibly repeated parts: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1). These are all the partitions of 5.

PartitionsRepeatedOdd(inf): Demonstrates that $\frac{1}{(q;q^2)_\infty}$ is the generating function for partitions with odd parts, possibly repeated. The low-order terms of the generated polynomial are 5*q**7 + 4*q**6 + 3*q**5 + 2*q**4 + 2*q**3 + q**2 + q + 1. Now for n = 5 there are three partitions with possibly repeated odd parts: (5), (3 1 1), (1 1 1 1 1); hence the coefficient of 3 on the q**5 term.

EulerOddEqualsDistinct (inf): Demonstrates that the generating function for all partitions using only odd parts produces the same number of partitions as the generating function for distinct parts. This is called "Euler's odd equals distinct theorem" in *Introduction to q-Analysis*.

Partitions with a Mixture of Parts

PartitionsRepeatedOddDistinctEven(n): Demonstrates that $\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}$ is the generating function for partitions with possibly repeated odd parts but distinct even parts. Note that this is a product of the generating function for repeated odd $\frac{1}{(q;q^2)_\infty}$ and distinct even $(-q^2;q^2)_\infty$.

The low-order terms of the generated polynomial are 12*q**7 + 9*q**6 + 6*q**5 + 4*q**4 + 3*q**3 + 2*q**2 + q + 1.

For example, for n = 5 observe that there are six partitions with possibly repeated odd parts but distinct even parts: (5), (4 1), (3 2), (3 1 1), (2 1 1 1), (1 1 1 1 1); hence the coefficient of the q**5 term is 6.

PartitionsNoMoreThanThree(n): Demonstrates that $\frac{(q^4;q^4)_\infty}{(q;q)_\infty}$ is the generating function for partitions in which no part is used more than three times.

The low-order terms of the generated polynomial are 12*q**7 + 9*q**6 + 6*q**5 + 4*q**4 + 3*q**3 + 2*q**2 + q + 1.

For n = 5, there are six partitions in which no part is used more than three times: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), which is reflected by the coefficient of 6 on the q**5 term.

CayleysTheorem(i,j): Demonstrates that $\binom{i+j}{i}_q$ is the generating function for partitions with at most i parts, each part at most j. This is different from all the other generating functions because it involves a *q*-binomial coefficient instead of a *qqn* function.

Recall that there are a total of seven partitions of 5: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1).

(i, j) = (3, 2): The single partition of 5 int0 at most three parts, each part at most 2, is (2 2 1). The polynomial is q**6 + q**5 + 2*q**4 + 2*q**3 + 2*q**2 + q + 1

(i, j) = (3, 3): There are three partitions of 5 int0 at most three parts, each part at most 3, are (3 2), (3 1 1), (2 2 1). The complete polynomial is q**9 + q**8 + 2*q**7 + 3*q**6 + 3*q**5 + 3*q**4 + 3*q**3 + 2*q**2 + q + 1.

(i, j) = (4, 2): The two partitions of 5 int0 at most four parts, each part at most 2, are (2 2 1), (2 1 1 1). The complete polynomial is q**8 + q**7 + 2*q**6 + 2*q**5 + 3*q**4 + 2*q**3 + 2*q**2 + q + 1.

This function calls the **qBinomialCoefficientFactorial(n,k)** function in the Inversions module.

PartitionsNotSingleton(n): Demonstrates that $\frac{(-q^3;q^3)_\infty}{(q^2;q^2)_\infty}$ is the generating function for partitions in which no part appears exactly once, and also the generating function for partitions in which all parts are congruent to 0, 2, 3, or 4 mod 6.

The low-order terms of the generated polynomial are 2*q**7 + 4*q**6 + q**5 + 2*q**4 + q**3 + q**2 + 1.

For n=5, recall that there are a total of seven partitions: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1). There is one partition in which no part appears exactly once: (1 1 1 1 1), and one partition in which all parts are congruent to 0, 2, 3, or 4 mod 6: (3 2). Hence, the coefficient of 1 on the q**5 term.

For n = 6, recall that there are 11 partitions: (6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1), (2 1 1 1 1), (1 1 1 1 1 1). There are four partitions in which no part appears exactly once: (3 3), (2 2 2), (2 2 1 1), (1 1 1 1 1), and four partitions in which all parts are congruent to 0, 2, 3, or 4 mod 6: (6), (4 2), (3 3), (2 2 2). Hence, the coefficient of 4 on the q**6 term.

Section III: Generating Functions–Finite

Now we study partition-generating functions that contain finite q-shifted factorials of the form $(q;q)_n$ and $\frac{1}{(q;q)_n}$. The generating functions with finite q-shifted factorials are shown in the table below.

#	Generating Function	Type of Partition
1	$\frac{1}{(q;q)_n} = \sum_{k=0}^{n} \frac{q^k}{(q;q)_k}$	most n parts
2	$\frac{q^{2k}}{(q;q^2)_\infty (q^2;q^2)_k}$	k is largest repeated part, or k even parts (Andrews-Deutsch theorem)
3	$\frac{q^{dk}}{(q^d;q^d)_k} \frac{(q^d;q^d)_\infty}{(q;q)_\infty}$	k is the largest part that occurs at least d times (Smoot-Yang theorem)
4	$\frac{(-q;q^2)_n}{(q^2;q^2)_n}$	at most n parts for which even parts may be repeated but odd parts are distinct (ee partition)
5	$q^n \frac{(-q;q^2)_n}{(q^2;q^2)_n}$	exactly n parts for which odd parts may be repeated but even parts are distinct (oo partition)
6	$\frac{1}{(x;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k}$	Euler identity: at most k parts
7	$\frac{1}{(xq;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k q^k}{(q;q)_k}$	Euler identity: exactly k parts
8	$(-x;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{(q;q)_k}$	Euler identity: at most k distinct parts
9	$\frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{k=0}^{\infty} \frac{(a;q)_k x^k}{(q;q)_k}$	q-binomial series

qqnInverse(A,B,k,inf): Cauchy proved that $\frac{1}{(x;q)_{k+1}} = \sum_{i=0}^{\infty} \binom{k+i}{\text{i}}_q x^i$, and I use this identity to approximate functions of the form $\frac{1}{(q;q)_k}$ and $\frac{1}{(q;q)_\infty}$. Using the Cauchy approximation, the Python function implements general functions of the form $\frac{1}{(A;B)_k} = \sum_{i=0}^{\infty} \binom{k-1+i}{\text{i}}_B A^i$.

When k is infinity, to decrease the calculation time, calls to qqnInverse(A,B,k,inf) are run concurrently (within the limits of your computer's CPU). This is controlled by the Python function qqnInverseControl(A, B, inf). This controller uses one less core than your full complement in order to provide some CPU for other programs that you may have open. However, it is better if you run these Python modules after closing other programs.

PartsAtMostN(n,inf): Infinite generating function #4 in the previous section, $\frac{1}{(q;q)_\infty}$, is the generating function for partitions in which some parts may be repeated, e.g. all partitions. The finite version, $\frac{1}{(q;q)_n}$, generates partitions with at most n parts. The identity, $\frac{1}{(q;q)_n}$ = $\sum_{k=0}^{n}\frac{q^k}{(q;q)_k}$, provides an alternative way of calculating the polynomial. For this and the other functions in this section, recall the partitions:

<u>Partitions of 1</u>: (1 1)

<u>Partitions of 2</u>: (2), (1 1)

<u>Partitions of 3</u>: (3), (2 1), (1 1 1)

<u>Partitions of 4</u>: (4), (3 1), (2 2), (2 1 1), (1 1 1 1)

<u>Partitions of 5</u>: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1)

<u>Partitions of 6</u>: (6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1), (2 1 1 1 1), (1 1 1 1 1 1).

For n = 2, the lower-order terms of the polynomial, 4*q**6 + 3*q**5 + 3*q**4 + 2*q**3 + 2*q**2 + q, tell us that there are two partitions of 2 and 3, 3 partitions of 4 and 5, and four partitions of 6 that have at most two parts. Looking at the partitions, we confirm that these partitions are (2), (1 1) for 2, (3), (2 1) for 3, (4), (3 1), (2 2) for 4, (5), (4 1), (3 2) for 5, and (6), (5 1), (4 2), (3 3) for 6.

For n = 3, the lower-order terms of the polynomial, 7*q**6 + 5*q**5 + 4*q**4 + 3*q**3 + 2*q**2 + q, tell us that there al the partitions for 2 and 3 have at most three partitions, 4 partitions of 4, five partitions of 5, and seven partitions of 6. Looking at the partitions, we confirm that these partitions are (4), (3 1), (2 2), (2 1 1) for 4, (5), (4 1), (3 2), (3 1 1), (2 2 1) for 5, and (6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (2 2 2) for 6.

AndrewsDeutsch(k,inf): The Andrews-Deutsch theorem proves that the function $\frac{q^{2k}}{(q;q^2)_\infty (q^2;q^2)_k}$ generates partitions both for k the largest repeated part and for k even parts.

For k = 2, the lower-order terms of the polynomial are 3*q**7 + 2*q**6 + q**5 + q**4.

First, this indicates that there should be no partitions of 1, 2, and 3, one partition of 4 and 5, two partitions of 6, and three partitions of 7 for which 2 is the largest repeated part. For 4, this partition is (2 2), for 5 it is (2 2 1), and for 6 they are (2 2 2), (2 2 1 1).

Second, it also indicates that there should be the same number of partitions for which there are k even parts. For 4, this partition is (2 2), for 5 it is (2 2 1), and for 6 they are (2 2 2), (2 2 1 1). These are the same partitions, but this is not true for all k as we see from k = 3.

For k = 3, the lower-order terms of the polynomial are 5*q**10 + 3*q**9 + 2*q**8 + q**7 + q**6. This indicates that there should be no partitions of 1, 2, 3, 4, and 5 for which 3 is the largest repeated part, which we can see is true. For 6 there should be one partition, and it is (3, 3).

SmootYang(k,d,inf): The Smoot-Yang theorem states that $\frac{q^{dk}}{(q^d;q^d)_k}\frac{(q^d;q^d)_\infty}{(q;q)_\infty}$ is the generating function for partitions: a) with exactly k parts divisible by d, and b) in which k is the largest part that occurs at least d times.

The infinite component of this generating function $\frac{(q^d;q^d)_\infty}{(q;q)_\infty}$ generates partitions with parts that are not divisible by d. The finite component $\frac{q^{dk}}{(q^d;q^d)_k}$ is a generating function for the k parts divisible by d. Multiplying the two components results in the generating function for partitions with exactly k parts divisible by d.

In addition to the polynomial generated by the complete generating function, the Python function prints the polynomial for the infinite component. For the examples, I used 10 as an approximation for infinity, which means that I can trust the resulting polynomial up to the q**10 term. I use k=3, d=3 and then k=2, d=2 to illustrate the concepts.

Infinite component: This component generates partitions with parts that are not divisible by d.

For example, for d = 3, the coefficients of the generated polynomial indicate the number of partitions with parts that are not divisible by 3. The low-order terms of the generated polynomial are 7*q**6 + 5*q**5 + 4*q**4 + 2*q**3 + 2*q**2 + q + 1.

For the 5*q**5 term: Recall that there are seven partitions of 5: **(5)**, **(4 1)**, (3 2), (3 1 1), **(2 2 1)**, **(2 1 1 1)**, **(1 1 1 1 1)**, and observe that the five bolded partitions have parts that are not divisible by 3, as indicated by the coefficient of 5 on the q**5 term.

For the 7*q**6 term: Recall that there 11 partitions of 6: (6), **(5 1)**, **(4 2)**, **(4 1 1)**, (3 3), (3 2 1), (3 1 1 1), **(2 2 2)**, **(2 2 1 1)**, **(2 1 1 1 1)**, **(1 1 1 1 1 1)**, and observe that seven of them have parts that are not divisible by 3, as indicated by the coefficient of 7 on the q**6 term.

For d = 2, the low-order terms of the generated polynomial are 4*q**6 + 3*q**5 + 2*q**4 + 2*q**3 + q**2 + q + 1. Among the seven partitions of 5 you can see that there are three that have parts that are not divisible by 2: (5), (3 1 1), and (1 1 1 1 1), as indicated by the coefficient of 3 on the q**5 term. Among the 11 partitions of 6 you can see that there are four that have parts that are not divisible by 2: (5 1), (3 3), (3 1 1 1), and (1 1 1 1 1 1) as indicated by the coefficient of 4 on the q**6 term.

Complete Generating Function: The complete function generates partitions: a) with exactly k parts divisible by d, and b) in which k is the largest part that occurs at least d times.

For example, for k = 3, d = 3, the coefficients of the generated polynomial indicate the number of partitions: a) with exactly 3 parts divisible by 3, and b) in which 3 is the largest part that occurs at least 3 times. The lower-order terms of the generated polynomial are q**10 + q**9. For the integer 9, the partition (3 3 3) satisfies a) and b).

For example, for k = 2, d = 2, the coefficients of the generated polynomial indicate the number of partitions: a) with 2 parts divisible by 2, and b) in which 2 is the largest part that occurs at least 2 times. The lower-order terms of the generated polynomial are 11*q**10 + 8*q**9 + 5*q**8 + 4*q**7 + 2*q**6 + q**5 + q**4. The partitions are (2 2) for 4; (2 2 1) for 5; (4 2) and (2 2 1 1) for 6; (4 2 1), (3 2 2), (2 2 2 1), and (2 2 1 1 1) for 7; and so on.

eePartitions(n): The function $\frac{(-q;q^2)_n}{(q^2;q^2)_n}$ generates partitions with at most n parts for which even parts may be repeated but odd parts are distinct. For n = 5, the low-order terms of the generated polynomial are 7*q**7 + 5*q**6 + 4*q**5 + 3*q**4 + 2*q**3 + q**2 + q + 1.

Partitions of 3: Of the three partitions (3), (2 1), (1 1 1), the first two are ee partitions, and therefore the coefficient of q**3 is 3.

Partitions of 4: Of the five partitions (4), (3 1), (2 2), (2 1 1), (1 1 1 1), only three are ee partitions: (4), (3, 1), (2, 2), and therefore the coefficient of q**4 is 3.

Partitions of 5: There are seven partitions of 5: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1). The ee partitions are (5), (4 1), (3 2), (2 2 1), and therefore the coefficient of q**5 is 4.

Partitions of 6: There are 11 partitions of 6: (6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1), (2 1 1 1 1), (1 1 1 1 1 1). The ee partitions are (6), (5 1), (4 2), (3 2 1), (2 2 2) , and therefore the coefficient of q**6 is 5.

ooPartitions(n): The function $q^n \frac{(-q;q^2)_n}{(q^2;q^2)_n}$ generates partitions with at exactly n parts for which odd parts may be repeated but repeated parts are distinct. For n = 3, the low-order terms of the generated polynomial are 3*q**7 + 2*q**6 + q**5 + q**4 + q**3. Certainly there are no partitions with exactly 3 parts for any integer less than 3.

Partitions of 3: Of the three partitions (1 1 1) is an oo partitions, and therefore the coefficient of q**3 is 1.

Partitions of 4: Of the five partitions only (2 1 1) is an oo partitions, and therefore the coefficient of q**4 is 1.

Partitions of 5: There are seven partitions of 5: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1). The 00 partition is (3 1 1), and therefore the coefficient of q**5 is 1.

Partitions of 6: There are 11 partitions of 6: (6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1), (2 1 1 1 1), (1 1 1 1 1 1). The oo partitions are (4 1 1), (3 2 1), and therefore the coefficient of q**6 is 2.

Three Euler Identities

These three identities introduce *x* into the generating function, which then produces coefficients of the x^k that are q-polynomials. These polynomials are generating functions for partitions with at most k parts, exactly k parts, or k distinct parts.

EulerIdentityAtMostK(inf): Euler provides an identity for the generating function $\frac{1}{(x;q)_\infty}$, which generates polyomials for partitions with at most k parts. The identity is $\frac{1}{(x;q)_\infty}$ = $\sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k}$. For example, using 7 to approximate infinity, we obtain polynomials that are accurate up to the sixth order term. Both sides produce a coefficient of x^4 with low-order terms 9*q**6 + 6*q**5 + 5*q**4 + 3*q**3 + 2*q**2 + q + 1. This indicates that there are nine partitions of 6 and six partitions of 5 with at most 4 parts. Referring to the partitions of 6 and 5 shown above, we can confirm that this is correct, namely (6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1) for 6 and (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1) for 5.

EulerIdentityExactlyK(inf): Euler provides an identity for the generating function $\frac{1}{(xq;q)_\infty}$, which generates polynomials for partitions with exactly k parts. The identity is $\frac{1}{(xq;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k q^k}{(q;q)_k}$. Again using 7 to approximate infinity, both sides produce a coefficient of x^3 with low-order terms 3*q**6 + 2*q**5 + q**4 + q**3. This indicates that there are three partitions of 6 and two partitions of 5 with exactly 3 parts. Referring to the partitions of 6 and 5 shown above, we can confirm that this is correct, namely (4 1 1), (3 2 1), (2 2 2) for 6 and (3 1 1), (2 2 1) for 5.

EulerIdentityAtMostKDistinct(inf): This partition theorem of Euler provides an identity for the generating function $(-x;q)_\infty$, which generates polynomials for partitions with at most k distinct parts. The identity is $(-x;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{(q;q)_k}$. Using 7 to approximate infinity, both sides produce a coefficient of x^3 with low-order terms 3*q**6 + 2*q**5 + q**4 + q**3. This indicates that there are three partitions of 6 and two partitions of 5 with at most 3 distinct parts. Referring to the partitions of 6 and 5 shown above, we can confirm that this is correct. For 6 we have (5 1), (4 2), (3 2 1) and for 5 we have (4 1), (3 2).

qBinomialSeries(inf): Euler developed slightly different versions of two of the identities above. In addition to the first identity $\frac{1}{(x;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k}$, he showed that $\frac{1}{(bx;q)_\infty} = \sum_{k=0}^{\infty} \frac{b^k x^k}{(q;q)_k}$. In addition to the third identity $(-x;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{(q;q)_k}$, Euler showed that $(ax;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} (-a)^k x^k}{(q;q)_k}$. These identities were combined by Cauchy and Jacobi into a q-binomial series (with b = 1) that is written very elegantly as $\frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{k=0}^{\infty} \frac{(a;q)_k x^k}{(q;q)_k}$. This is a very important identity, and is also a theorem in Chapter 1 of Berndt's *Number theory in the Spirit of Ramanujan*. A sample of results demonstrates the equality:

The low-order terms of the coefficient of x**3 for the left-side are 7*q**7 + 7*q**6 + 5*q**5 + 4*q**4 + 3*q**3 + 2*q**2 + q + 1 and for the right side they are 7*q**7 + 7*q**6 + 5*q**5 + 4*q**4 + 3*q**3 + 2*q**2 + q + 1.

The low-order terms of the coefficient of x**2 for the left-side are 3*q**7 + 4*q**6 + 3*q**5 + 3*q**4 + 2*q**3 + 2*q**2 + q + 1 and for the right side 3*q**7 + 4*q**6 + 3*q**5 + 3*q**4 + 2*q**3 + 2*q**2 + q + 1.

