Connection between Inversions and *q*-Binomial Coefficients

by Alan Mehlenbacher

This article explains concepts for binary inversions, partition inversions, q-binomial coefficients, q-multinomial coefficients, and their connections.

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapter 2 of *An Introduction to q-analysis* by Warren P. Johnson.

1 Binary Sequence Inversions

1.1 Permutations and Inversions of Binary Sequences

For the binomial coefficient, we use permutations of 0's and 1's, although we could use any two integers. The tables below show binomial permutations for k 0's and n-k 1's, and inversions of these permutations. For example, for n=4, k=2 there are six permutations and a total of 12 inversions; for n=5, k=2 there are a total of 30 inversions; and for n=6, k=3 there are a total of 90 inversions.

Table 1: *n*=4, *k*=2

Permutations	Inv
(1, 1, 0, 0)	4
(1, 0, 1, 0)	3
(1, 0, 0, 1)	2
(0, 1, 1, 0)	2
(0, 1, 0, 1)	1
(0, 0, 1, 1)	0

Table 2: *n*=5, *k*=2

Permutations	Inv
(1, 1, 1, 0, 0)	6
(1, 1, 0, 1, 0)	5
(1, 0, 1, 1, 0)	4
(1, 1, 0, 0, 1)	4
(1, 0, 1, 0, 1)	3
(0, 1, 1, 1, 0)	3
(0, 1, 1, 0, 1)	2
(1, 0, 0, 1, 1)	2
(0, 1, 0, 1, 1)	1
(0, 0, 1, 1, 1)	0

Table 3: *n*=6, *k*=3

Permutations	Inv	Permutations	Inv
(1, 1, 1, 0, 0, 0)	9	(0, 1, 1, 0, 0, 1)	4
(1, 1, 0, 1, 0, 0)	8	(0, 1, 0, 1, 1, 0)	4
(1, 1, 0, 0, 1, 0)	7	(1, 0, 0, 1, 0, 1)	4
(1, 0, 1, 1, 0, 0)	7	(0, 1, 0, 1, 0, 1)	3
(1, 0, 1, 0, 1, 0)	6	(1, 0, 0, 0, 1, 1)	3
(1, 1, 0, 0, 0, 1)	6	(0, 0, 1, 1, 1, 0)	3
(0, 1, 1, 1, 0, 0)	6	(0, 1, 0, 0, 1, 1)	2
(0, 1, 1, 0, 1, 0)	5	(0, 0, 1, 1, 0, 1)	2
(1, 0, 1, 0, 0, 1)	5	(0, 0, 1, 0, 1, 1)	1
(1, 0, 0, 1, 1, 0)	5	(0, 0, 0, 1, 1, 1)	0

Table 4 on the next page shows the permutations and inversions for n=9, k=4, for which there are a total of 1,260 inversions.

Table 4: *n*=9, *k*=4							
Permutations	**Inv**	**Permutations**	**Inv**	**Permutations**	**Inv**	**Permutations**	**Inv**
(1,1,1,1,1,0,0,0,0)	20	(1,1,1,0,0,0,1,0,1)	13	(0,1,1,0,1,0,1,1,0)	10	(0,0,1,1,0,1,1,1,0)	7
(1,1,1,1,0,1,0,0,0)	19	(0,1,1,1,1,0,0,1,0)	13	(1,1,0,0,1,0,0,1,1)	10	(1,0,0,0,1,1,1,0,1)	7
(1,1,1,0,1,1,0,0,0)	18	(1,1,0,1,0,0,1,1,0)	13	(0,1,1,0,1,1,0,0,1)	10	(0,1,0,1,0,1,1,0,1)	7
(1,1,1,1,0,0,1,0,0)	18	(1,0,1,1,0,0,1,1,0)	12	(0,1,0,1,1,1,0,1,0)	10	(0,1,0,0,1,1,1,1,0)	7
(1,1,1,0,1,0,1,0,0)	17	(1,0,0,1,1,1,1,0,0)	12	(1,1,0,0,0,1,1,0,1)	10	(0,0,1,1,0,1,1,0,1)	6
(1,1,1,1,0,0,0,1,0)	17	(1,0,1,1,0,1,0,0,1)	12	(1,0,0,1,1,0,1,0,1)	9	(1,0,0,0,1,1,0,1,1)	6
(1,1,0,1,1,1,0,0,0)	17	(1,0,1,0,1,1,0,1,0)	12	(0,1,1,1,0,0,0,1,1)	9	(1,0,0,1,0,0,1,1,1)	6
(1,1,1,0,0,1,1,0,0)	16	(0,1,1,0,1,1,1,0,0)	12	(1,0,1,0,0,1,1,0,1)	9	(0,1,0,1,0,1,0,1,1)	6
(1,1,0,1,1,0,1,0,0)	16	(1,1,1,0,0,0,0,1,1)	12	(0,1,1,0,1,0,1,0,1)	9	(0,0,1,0,1,1,1,1,0)	6
(1,1,1,1,0,0,0,0,1)	16	(0,1,1,1,1,0,0,0,1)	12	(0,0,1,1,1,1,0,1,0)	9	(0,1,1,0,0,0,1,1,1)	6
(1,0,1,1,1,1,0,0,0)	16	(0,1,1,1,0,1,0,1,0)	12	(1,0,0,1,0,1,1,1,0)	9	(0,1,0,0,1,1,1,0,1)	6
(1,1,1,0,1,0,0,1,0)	16	(1,1,0,1,0,0,1,0,1)	12	(0,1,0,1,1,0,1,1,0)	9	(0,0,1,1,1,0,0,1,1)	6
(1,1,1,0,1,0,0,0,1)	15	(1,1,0,0,1,0,1,1,0)	12	(0,1,0,1,1,1,0,0,1)	9	(0,0,1,1,0,1,0,1,1)	5
(1,1,1,0,0,1,0,1,0)	15	(1,1,0,0,1,1,0,0,1)	12	(0,1,1,0,0,1,1,1,0)	9	(0,0,1,0,1,1,1,0,1)	5
(1,1,0,1,1,0,0,1,0)	15	(1,1,0,0,0,1,1,1,0)	11	(1,1,0,0,0,1,0,1,1)	9	(0,0,0,1,1,1,1,1,0)	5
(1,0,1,1,1,0,1,0,0)	15	(1,0,1,1,0,0,1,0,1)	11	(1,0,1,0,1,0,0,1,1)	9	(1,0,0,0,1,0,1,1,1)	5
(0,1,1,1,1,1,0,0,0)	15	(1,0,1,0,1,0,1,1,0)	11	(1,0,0,1,1,0,0,1,1)	8	(0,1,0,0,1,1,0,1,1)	5
(1,1,0,1,0,1,1,0,0)	15	(1,0,1,0,1,1,0,0,1)	11	(1,0,1,0,0,1,0,1,1)	8	(0,1,0,1,0,0,1,1,1)	5
(1,1,1,0,0,0,1,1,0)	14	(1,0,0,1,1,1,0,1,0)	11	(0,1,1,0,1,0,0,1,1)	8	(0,0,1,0,1,1,0,1,1)	4
(1,1,1,0,0,1,0,0,1)	14	(0,1,1,1,0,0,1,1,0)	11	(0,0,1,1,1,0,1,1,0)	8	(0,0,0,1,1,1,1,0,1)	4
(1,1,0,1,1,0,0,0,1)	14	(0,1,0,1,1,1,1,0,0)	11	(0,0,1,1,1,1,0,0,1)	8	(0,0,1,1,0,0,1,1,1)	4
(1,1,0,1,0,1,0,1,0)	14	(1,1,0,1,0,0,0,1,1)	11	(1,0,0,1,0,1,1,0,1)	8	(1,0,0,0,0,1,1,1,1)	4
(1,0,1,1,0,1,1,0,0)	14	(0,1,1,1,0,1,0,0,1)	11	(0,1,0,1,1,0,1,0,1)	8	(0,1,0,0,1,0,1,1,1)	4
(1,0,1,1,1,0,0,1,0)	14	(0,1,1,0,1,1,0,1,0)	11	(0,1,1,0,0,1,1,0,1)	8	(0,0,0,1,1,1,0,1,1)	3
(0,1,1,1,1,0,1,0,0)	14	(1,1,0,0,1,0,1,0,1)	11	(1,0,0,0,1,1,1,1,0)	8	(0,0,1,0,1,0,1,1,1)	3
(1,1,0,0,1,1,1,0,0)	14	(1,0,1,1,0,0,0,1,1)	10	(0,1,0,1,0,1,1,1,0)	8	(0,1,0,0,0,1,1,1,1)	3
(1,1,0,1,0,1,0,0,1)	13	(1,0,1,0,1,0,1,0,1)	10	(1,1,0,0,0,0,1,1,1)	8	(0,0,1,0,0,1,1,1,1)	2
(1,1,0,0,1,1,0,1,0)	13	(0,0,1,1,1,1,1,0,0)	10	(0,0,1,1,1,0,1,0,1)	7	(0,0,0,1,1,0,1,1,1)	2
(1,0,1,0,1,1,1,0,0)	13	(1,0,0,1,1,0,1,1,0)	10	(1,0,0,1,0,1,0,1,1)	7	(0,0,0,1,0,1,1,1,1)	1
(1,0,1,1,1,0,0,0,1)	13	(1,0,0,1,1,1,0,0,1)	10	(0,1,0,1,1,0,0,1,1)	7	(0,0,0,0,1,1,1,1,1)	0
(1,0,1,1,0,1,0,1,0)	13	(1,0,1,0,0,1,1,1,0)	10	(1,0,1,0,0,0,1,1,1)	7		
(0,1,1,1,0,1,1,0,0)	13	(0,1,1,1,0,0,1,0,1)	10	(0,1,1,0,0,1,0,1,1)	7		

1.2 *q*-Binomial Coefficient

For sequences of *k* 0's and *n*-*k* 1's or *k* x's and *n*-*k* y's, the *q*-binomial coefficient is $\binom{n}{k}_q = \frac{n!_q}{k!_q(n-k)!_q}$. For example, for $n = 4$ and $k = 2$, the *q*-binomial coefficient is $q^4 + q^3 + 2q^2 + q + 1$.

Setting q = 1, we have $\binom{4}{2}_1 = \frac{4!_1}{2!_1(2)!_1} = 6 = \binom{4}{2}$.

1.3 Inversions and *q*-Binomial Coefficient Connection

For inversions of binary sequences, the Fundamental Property of *q*-Binomial Coefficients shows that $\binom{n}{k}_q = \sum_{\sigma \in S(k,n-k)} q^{inv\ \sigma}$, where *S* is the set of all permutations of *k* 0's and *n*-k 1's. This identity may have been known by Gauss in 1808, but the first definitive origin appears to be MacMahon in 1913.

The fact that the polynomial from the *q*-binomial coefficient for *n* and *k* is the same as the polynomial from the sum of *q* raised to the power of the number of inversions for each permutation is an amazing result.

You can prove this by induction, but it is very useful to observe that this is the case by running the Python function using several examples. Using Python code for both sides of the Fundamental Property of *q*-Binomial Coefficients verifies the identity of the polynomials, as shown in Table 5 on the next page.

The *q*-binomial coefficient for (4, 2) is q**4 + q**3 + 2*q**2 + q + 1, which tells us that there is 1 permutation with four inversions, 1 with three inversions, 2 with two inversions, 1 with one inversion, and 1 with no inversions, for a total of 12 inversions. This is the same result shown in Table 1 without the need to generate all the permutations and inversions.

Sampling some results from Table 5, we have:

> for (5, 2) the coefficient of q**4 is 2, i.e., there are 2 permutations with four inversions, which agrees with Table 2,
>
> for (6, 3) the coefficient of q**6 is 3, i.e., there are 3 permutations with six inversions, which agrees with Table 3, and
>
> for (9, 2) the coefficient of q**14 is 8, i.e., there are 8 permutations with 14 inversions, which agrees with Table 4.

n,k	**Table 5: Fundamental Property of *q*-Binomial Coefficients**	
5, 2	$\binom{n}{k}_q$	q**6 + q**5 + 2*q**4 + 2*q**3 + 2*q**2 + q + 1
	$\sum_{\sigma\in S(k,n-k)} q^{inv\ \sigma}$	q**6 + q**5 + 2*q**4 + 2*q**3 + 2*q**2 + q + 1
6, 3	$\binom{n}{k}_q$	q**9 + q**8 + 2*q**7 + 3*q**6 + 3*q**5 + 3*q**4 + 3*q**3 + 2*q**2 + q + 1
	$\sum_{\sigma\in S(k,n-k)} q^{inv\ \sigma}$	q**9 + q**8 + 2*q**7 + 3*q**6 + 3*q**5 + 3*q**4 + 3*q**3 + 2*q**2 + q + 1
9, 4	$\binom{n}{k}_q$	q**20 + q**19 + 2*q**18 + 3*q**17 + 5*q**16 + 6*q**15 + 8*q**14 + 9*q**13 + 11*q**12 + 11*q**11 + 12*q**10 + 11*q**9 + 11*q**8 + 9*q**7 + 8*q**6 + 6*q**5 + 5*q**4 + 3*q**3 + 2*q**2 + q + 1
	$\sum_{\sigma\in S(k,n-k)} q^{inv\ \sigma}$	q**20 + q**19 + 2*q**18 + 3*q**17 + 5*q**16 + 6*q**15 + 8*q**14 + 9*q**13 + 11*q**12 + 11*q**11 + 12*q**10 + 11*q**9 + 11*q**8 + 9*q**7 + 8*q**6 + 6*q**5 + 5*q**4 + 3*q**3 + 2*q**2 + q + 1

2 Partition Inversions

2.1 Binary Partitions and Inversions between Partitions

Now we look at binary partitions of an integer sequence, and the inversions between the sets of each partition. A binary partition (*n*, *k*) is the set of all subsets of size *k* and *n*-*k*. Examples of binary partitions and their inversions are shown in Tables 6-9.

Table 6: (3, 2)

Partitions	Inv
{2,3} {1}	2
{1,3} {2}	1
{1,2} {3}	0

Table 7: (4, 2)

Partitions	Inv
{3, 4} {1, 2}	4
{2, 4} {1, 3}	3
{1, 4} {2, 3}	2
{2, 3} {1, 4}	2
{1, 3} {2, 4}	1
{1, 2} {3, 4}	0

Table 8: (5, 2)

Partitions	Inv
{4, 5} {1, 2, 3}	6
{3, 5} {1, 2, 4}	5
{2, 5} {1, 3, 4}	4
{3, 4} {1, 2, 5}	4
{1, 5} {2, 3, 4}	3
{2, 4} {1, 3, 5}	3
{1, 4} {2, 3, 5}	2
{2, 3} {1, 4, 5}	2
{1, 3} {2, 4, 5}	1
{1, 2} {3, 4, 5}	0

Table 9: (6, 4)

Partitions	Inv
{3, 4, 5, 6} {1, 2}	8
{2, 4, 5, 6} {1, 3}	7
{1, 4, 5, 6} {2, 3}	6
{2, 3, 5, 6} {1, 4}	6
{1, 3, 5, 6} {2, 4}	5
{2, 3, 4, 6} {1, 5}	5
{1, 2, 5, 6} {3, 4}	4
{1, 3, 4, 6} {2, 5}	4
{2, 3, 4, 5} {1, 6}	4
{1, 2, 4, 6} {3, 5}	3
{1, 3, 4, 5} {2, 6}	3
{1, 2, 3, 6} {4, 5}	2
{1, 2, 4, 5} {3, 6}	2
{1, 2, 3, 5} {4, 6}	1
{1, 2, 3, 4} {5, 6}	0

2.2 Partition Inversions and q-Binomial Coefficient Connection

The amazing thing about these partition inversions is that they connect with the q-binomial coefficient in exactly the same way as the inversions of binary sequences. We have

$\binom{n}{k}_q = \sum_{t \in T(k,n-k)} q^{inv\ t}$, where t is a partition in the set of partitions $T(k, n-k)$.

Using Python code for both sides of this equation verifies the identity of the polynomials, as shown in Table 10.

For (3, 2), the coefficients are all 1, which agrees with the number of inversions in Table 6. For (4, 2) Table 10 shows that there is 1 partition with four inversions, 1 with three inversions, 2 with two inversions, 1 with one inversion, and 1 with no inversions—again agreeing with Table 7.

Sampling more results from Table 10, we find for (5, 2) the coefficient of q**4 is 2, i.e., there are 2 partitions with four inversions, which agrees with Table 8. For (6, 4) the coefficient of q**6 is 2, i.e., there are 2 partitions with six inversions, which agrees with Table 9.

n,k	**Table 10: *q*-Binomial Coefficients and Partition Inversions**	
3, 2	$\binom{n}{k}_q$	q**2 + q + 1
	$\sum_{t \in T(k,n-k)} q^{inv\ t}$	q**2 + q + 1
4, 2	$\binom{n}{k}_q$	q**4 + q**3 + 2*q**2 + q + 1
	$\sum_{t \in T(k,n-k)} q^{inv\ t}$	q**4 + q**3 + 2*q**2 + q + 1
5, 2	$\binom{n}{k}_q$	q**6 + q**5 + 2*q**4 + 2*q**3 + 2*q**2 + q + 1
	$\sum_{t \in T(k,n-k)} q^{inv\ t}$	q**6 + q**5 + 2*q**4 + 2*q**3 + 2*q**2 + q + 1
6, 4	$\binom{n}{k}_q$	q**8 + q**7 + 2*q**6 + 2*q**5 + 3*q**4 + 2*q**3 + 2*q**2 + q + 1
	$\sum_{t \in T(k,n-k)} q^{inv\ t}$	q**8 + q**7 + 2*q**6 + 2*q**5 + 3*q**4 + 2*q**3 + 2*q**2 + q + 1

3 Multinomial Sequence Inversions

3.1 Permutations of Multinomial Sequences

Binomial sequences of k 0's and n-k 1's, can be generalized to multinomial sequences of k_1 1's, k_2 2's, …, k_m m's. Table 11 shows some multinomial sequences.

Table 11:	
(k_1, k_2, k_3, k_4)	**Multinomial Sequence**
(2, 2, 0, 0)	[1, 1, 2, 2]
(2, 2, 2, 0)	[1, 1, 2, 2, 3, 3]
(2, 2, 2, 2)	[1, 1, 2, 2, 3, 3, 4, 4]

3.2 Inversions and *q*-Multinomial Coefficient Connection

The Fundamental Property of q-Multinomial Coefficients $\binom{n}{k_1,\ldots,k_m}_q = \sum_{\sigma\in S(k_1,\ldots,k_m)} q^{inv\ \sigma}$, a generalization of the Fundamental Property of q-Binomial Coefficients, was proven by MacMahon in 1913. Using Python code verifies the identity of the polynomials, as shown in Table 12. Note that the results for (3, 2, 0, 0) and (3, 3, 0, 0) are the same as the binomial results in Section 1 for (5, 2) and (6, 3).

Table 12		
(k_1, k_2, k_3, k_4)	**Fundamental Property of *q*-Multinomial Coefficients**	
(2, 2, 0, 0)	$\binom{n}{k_1,\ldots,k_m}_q$	q**4 + q**3 + 2*q**2 + q + 1
	$\sum_{\sigma\in S(k_1,\ldots,k_m)} q^{inv\ \sigma}$	q**4 + q**3 + 2*q**2 + q + 1
(3, 2, 0, 0)	$\binom{n}{k_1,\ldots,k_m}_q$	q**6 + q**5 + 2*q**4 + 2*q**3 + 2*q**2 + q + 1
	$\sum_{\sigma\in S(k_1,\ldots,k_m)} q^{inv\ \sigma}$	q**6 + q**5 + 2*q**4 + 2*q**3 + 2*q**2 + q + 1
(3, 3, 0, 0)	$\binom{n}{k_1,\ldots,k_m}_q$	q**9 + q**8 + 2*q**7 + 3*q**6 + 3*q**5 + 3*q**4 + 3*q**3 + 2*q**2 + q + 1
	$\sum_{\sigma\in S(k_1,\ldots,k_m)} q^{inv\ \sigma}$	q**9 + q**8 + 2*q**7 + 3*q**6 + 3*q**5 + 3*q**4 + 3*q**3 + 2*q**2 + q + 1
(2, 2, 2, 2)	$\binom{n}{k_1,\ldots,k_m}_q$	q**12 + 2*q**11 + 5*q**10 + 7*q**9 + 11*q**8 + 12*q**7 + 14*q**6 + 12*q**5 + 11*q**4 + 7*q**3 + 5*q**2 + 2*q + 1
	$\sum_{\sigma\in S(k_1,\ldots,k_m)} q^{inv\ \sigma}$	q**12 + 2*q**11 + 5*q**10 + 7*q**9 + 11*q**8 + 12*q**7 + 14*q**6 + 12*q**5 + 11*q**4 + 7*q**3 + 5*q**2 + 2*q + 1

