# Falls

A fall is a relationship between adjacent elements of a permutation. It is the position of an element that is larger than its next element. The sum of the falls is called the "major index" of the permutation and, amazingly, the major index can be used in the same way as the number of inversions.

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapter 9 of *An Introduction to q-analysis* by Warren P. Johnson.

## 1 Falls and Major Indexes

The articles in the Inversions topic used the total inversions of permutations for $q$-factorials and $q$-binomial coefficients. Falls can be used for the same purpose as inversions.

Recall that an inversion of a permutation is a pair of elements in decreasing order, but the elements are not necessarily adjacent. We count the number of times this occurs in the permutation; thus (3, 1, 2) and (3, 2, 1) have two inversions each.

Now, for falls we consider adjacent elements, record the position of an element that is larger than its next element, and call it a *fall*. Thus, (3, 1, 2) has a fall at position 1 and (3, 2, 1) has two falls at positions 1 and 2. Then, instead of counting, we sum the falls and call the sum of the falls the "major index," denoted *maj*. For (3, 1, 2) and (3, 2, 1) we have $maj = 1$ and $maj = 1 + 2 = 3$, respectively.

Table 1 shows the falls (in red) and major indexes for all the permutations of {1, 2, 3}.

**Table 1: Falls and Major Indexes for Permutations of {1, 2, 3}**

| Permutation | Falls | Major Index | Falls Explanation |
|---|---|---|---|
| (1, 2, 3) | | 0 | No larger element |
| (1, 3 ,2) | 2 | 2 | 3 in position 2 |
| (2, 1, 3) | 1 | 1 | 2 in position 1 |
| (2, 3, 1) | 2 | 2 | 3 in position 2 |
| (3, 1, 2) | 1 | 1 | 3 in position 1 |
| (3, 2, 1) | 1, 2 | 3 | 3 in position 1, 2 in position 2 |

 

Table 2 shows major indexes for the first five integers. For $n = 5$, the number major indexes of 6 are highlighted in blue and those with 9 are highlighted in red.

**Table 2: Major Indexes for *n* = 1 to 5**

| ***n*** | **Python Output** |
|---|---|
| 1 | [0] |
| 2 | [0, 1] |
| 3 | [0, 2, 1, 2, 1, 3], as shown in Table 1. |
| 4 | [0, 3, 2, 3, 2, 5, 1, 4, 2, 3, 2, 5, 1, 4, 3, 4, 2, 5, 1, 4, 3, 4, 3, 6] |
| 5 | [0, 4, 3, 4, 3, 7, 2, 6, 3, 4, 3, 7, 2, 6, 5, 6, 3, 7, 2, 6, 5, 6, 5, 9, 1, 5, 4, 5, 4, 8, 2, 6, 3, 4, 3, 7, 2, 6, 5, 6, 3, 7, 2, 6, 5, 6, 5, 9, 1, 5, 4, 5, 4, 8, 3, 7, 4, 5, 4, 8, 2, 6, 5, 6, 3, 7, 2, 6, 5, 6, 5, 9, 1, 5, 4, 5, 4, 8, 3, 7, 4, 5, 4, 8, 3, 7, 6, 7, 4, 8, 2, 6, 5, 6, 5, 9, 1, 5, 4, 5, 4, 8, 3, 7, 4, 5, 4, 8, 3, 7, 6, 7, 4, 8, 3, 7, 6, 7, 6, 10] |

## 2 Connection between *q*-Factorial and Major Indexes

Recall from the Inversions topic that $n!_q = [1]_q[2]_q...[n]_q$. For example, for $3!_q$ we calculate (q + 1)*(q**2 + q + 1) = q**3 + 2*q**2 + 2*q + 1.

Also, for inversions of integer sequences, Rodrigues' theorem (see the article Connections between Inversions and $q$-Factorials) shows that $n!_q = \sum_{p \in P} q^{inv\ p}$, where *inv p* is the number of inversions for permutation *p* and *P* is the set of all permutations of $\{1,\ldots,n\}$.

Now we have the same type of identity using the major index instead of inversions: $n!_q = \sum_{p \in P} q^{maj\ p}$. Table 3 shows some Python output for this identity.

 

| **Table 3:** $n!_q = \sum_{p \in P} q^{inv\ p}$ | |
|---|---|
| ***n*** | **Python Output** |
| 2 | *n*!q: q + 1<br>SUM(q^maj(p)): q + 1 |
| 3 | *n*!q: q**3 + 2*q**2 + 2*q + 1<br>SUM(q^maj(p)): q**3 + 2*q**2 + 2*q + 1 |
| 4 | *n*!q: q**6 + 3*q**5 + 5*q**4 + 6*q**3 + 5*q**2 + 3*q + 1<br>SUM(q^maj(p)): q**6 + 3*q**5 + 5*q**4 + 6*q**3 + 5*q**2 + 3*q + 1 |
| 5 | *n*!q: q**10 + 4*q**9 + 9*q**8 + 15*q**7 + 20*q**6 + 22*q**5 + 20*q**4 + 15*q**3 + 9*q**2 + 4*q + 1<br>SUM(q^maj(p)): q**10 + 4*q**9 + 9*q**8 + 15*q**7 + 20*q**6 + 22*q**5 + 20*q**4 + 15*q**3 + 9*q**2 + 4*q + 1 |

Note that the coefficients for each *q**k* are the number of permutations with *maj* = k. For example, when $n = 5$, there are four permutations with *maj* = 9 (highlighted with red in Tables 2 and 3) and 20 permutations with *maj* = 6 (highlighted with blue in Tables 2 and 3).

# 3 Connection between Major Index and *q*-Binomial Coefficient

Recall from the Inversions topic that $\sum_{\sigma \in S(k,n-k)} q^{inv\ \sigma} = \binom{n}{k}_q$, where *S* is the set of all permutations of *k* 0's and *n*-k 1's (see the article Connection between Inversions and *q*-Binomial Coefficients). The major index can be used instead of inversions so that $\sum_{\sigma \in S(k,n-k)} q^{maj\ \sigma} = \binom{n}{k}_q$. Using an alternative notation for the binomial coefficient, we have the equivalent expression $\sum_{\sigma \in S(a,b)} q^{maj\ \sigma} = \binom{a+b}{a}_q$, where *S* is the set of all permutations of *a* 0's and *b* 1's. Table 4 shows some Python output for this identity.

| **Table 4:** $\sum_{\sigma \in S(a,b)} q^{maj\ \sigma} = \binom{a+b}{a}_q$ | | |
|---|---|---|
| **a** | **b** | **Python Output** |
| 2 | 2 | maj: q**4 + q**3 + 2*q**2 + q + 1<br>qbc: q**4 + q**3 + 2*q**2 + q + 1 |
| 4 | 2 | maj: q**8 + q**7 + 2*q**6 + 2*q**5 + 3*q**4 + 2*q**3 + 2*q**2 + q + 1<br>qbc: q**8 + q**7 + 2*q**6 + 2*q**5 + 3*q**4 + 2*q**3 + 2*q**2 + q + 1 |

 