Sum of Squares

In this article, we use Cauchy's second ${}_1\psi_1$ identity to derive a polynomial for which the coefficient of q^m is the number of ways of writing *m* as a sum of two squares, and we use Cauchy's first ${}_1\psi_1$ identity to derive a polynomial for which the coefficient of q^m is the number of ways of writing *m* as a sum of four squares.

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapter 7 of *An Introduction to q-analysis* by Warren P. Johnson.

1 Sum of Two Squares

It is amazing that we can take Cauchy's second ${}_1\psi_1$ identity $\sum_{n=-\infty}^{\infty}\frac{2x^n q^{2n}}{1+q^{2n}} = \frac{(-xq;q^2)_\infty\left(-\frac{q}{x};q^2\right)_\infty (q^2;q^2)_\infty^2}{(xq;q^2)_\infty\left(\frac{q}{x};q^2\right)_\infty \left(-q^2;q^2\right)_\infty^2}$ and, using some clever rewriting, produce a simple q-polynomial expression for the sum of two squares. Then with more rewriting, we produce a non-q expression using divisors.

1.1 Derivation of the q-Polynomial expression

Set $x = 1$ to produce $\sum_{n=-\infty}^{\infty}\frac{2q^{2n}}{1+q^{2n}} = \frac{(-q;q^2)_\infty\left(-q;q^2\right)_\infty (q^2;q^2)_\infty^2}{(q;q^2)_\infty\left(q;q^2\right)_\infty \left(-q^2;q^2\right)_\infty^2}$. The connection with the sum of two squares begins to emerge. The product term can be written $\left(\frac{\left(-q;q^2\right)_\infty\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty\left(-q^2;q^2\right)_\infty}\right)^2$, which tells us that $\sum_{n=-\infty}^{\infty}\frac{2q^{2n}}{1+q^{2n}}$ is a perfect square!

This product term can be further rewritten using Euler's odd-equals-distinct theorem $(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}$ from the article Partitions with Distinct and Repeated Parts and Jacobi's triple product identity (with z = 1) $(-q;q^2)_\infty \left(-q;q^2\right)_\infty (q^2;q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2}$ from the article Triple Product Identities. This results in the considerably simplified expression $2\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1+q^{2n}} = \left(\sum_{n=-\infty}^{\infty} q^{n^2} \right)^2$.

Since the right side equals $\left(\sum_{j=-\infty}^{\infty} q^{j^2} \right)\left(\sum_{k=-\infty}^{\infty} q^{k^2} \right) = \sum_{j,k=-\infty}^{\infty} q^{j^2+k^2}$, we have $2\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1+q^{2n}} = \sum_{j,k=-\infty}^{\infty} q^{j^2+k^2}$. This tells us that the coefficient of q^m on the left side is the number of ways of writing *m* as a sum of two squares. Because of symmetry, we can simplify $\sum_{n=-\infty}^{\infty} \frac{2q^{2n}}{1+q^{2n}} = 1+4\sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}}$ and conclude that $\sum_{m=0}^{\infty} S_2(m)q^m = 1+4\sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}}$, where $S_2(m)$ is the number of ways of writing *m* as a sum of two squares. It is important to note that this allows for both positive and negative numbers and permutations.

The first 25 terms of the resulting polynomial are:

12*q**25 + 8*q**20 + 4*q**18 + 8*q**17 + 4*q**16 + 8*q**13 + 8*q**10 + 4*q**9 + 4*q**8 + 8*q**5 + 4*q**4 + 4*q**2 + 4*q + 1.

This polynomial demonstrates that there are four ways to write 1, 2, and 4 as a sum of two squares. Checking this, we find:

for 1: 1^2+0^2, 0^2+1^2, $(-1)^2+0^2$, $0^2+(-1)^2$;

for 2: 1^2+1^2, 1^2+1^2, $1^2+(-1)^2$, $(-1)^2+1^2$; and

for 4: 2^2+0^2, 0^2+2^2, $(-2)^2+0^2$, $0^2+(-2)^2$.

1.2 Derivation of the non-q Divisor Expression

The right side of the previous expression $1+4\sum_{n=1}^{\infty}\frac{q^{2n}}{1+q^{2n}}$ can be written $1+4\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(q^{n(4k+1)}-q^{n(4k+3)}\right)$, and simplified to $1+4\sum_{n=1}^{\infty}\left(d_{1,4}(m)-d_{3,4}(m)\right)q^m$, where $d_{1,4}(m)$ is the number of divisors of *m* that are congruent to 1 mod 4 and $d_{3,4}(m)$ is the number of divisors of *m* that are congruent to 3 mod 4 (see Section 7.2 of Johnson). Now we have $\sum_{m=0}^{\infty}S_2(m)q^m =$ $1+4\sum_{n=1}^{\infty}\left(d_{1,4}(m)-d_{3,4}(m)\right)q^m$.

However, note that in this expression, we no longer need the polynomials! We can simply write $S_2(m) = 4\left(d_{1,4}(m)-d_{3,4}(m)\right)$, which is known as Jacobi's two square theorem. This function calculates $S_2(m)$ for $m = 0$ to n.

For *m* up to 200, Table 1 shows the coefficients that represent the number of ways to write *m* as a sum of two squares. In this range, $S_2(m)$ ranges from 0 to 16, and for most *m* we cannot write *m* as a sum of two squares ($S_2(m) = 0$).

Table 1: Number of ways to write m as a sum of two squares

m	$S_2(m)$	m	$S_2(m)$	m	$S_2(m)$	m	$S_2(m)$	m	$S_2(m)$
1	4	41	8	81	4	121	4	161	0
2	4	42	0	82	8	122	8	162	4
3	0	43	0	83	0	123	0	163	0
4	4	44	0	84	0	124	0	164	8
5	8	45	8	85	16	125	16	165	0
6	0	46	0	86	0	126	0	166	0
7	0	47	0	87	0	127	0	167	0
8	4	48	0	88	0	128	4	168	0
9	4	49	4	89	8	129	0	169	12
10	8	50	12	90	8	130	16	170	16
11	0	51	0	91	0	131	0	171	0
12	0	52	8	92	0	132	0	172	0
13	8	53	8	93	0	133	0	173	8
14	0	54	0	94	0	134	0	174	0
15	0	55	0	95	0	135	0	175	0
16	4	56	0	96	0	136	8	176	0
17	8	57	0	97	8	137	8	177	0
18	4	58	8	98	4	138	0	178	8
19	0	59	0	99	0	139	0	179	0
20	8	60	0	100	12	140	0	180	8
21	0	61	8	101	8	141	0	181	8
22	0	62	0	102	0	142	0	182	0
23	0	63	0	103	0	143	0	183	0
24	0	64	4	104	8	144	4	184	0
25	12	65	16	105	0	145	16	185	16
26	8	66	0	106	8	146	8	186	0
27	0	67	0	107	0	147	0	187	0
28	0	68	8	108	0	148	8	188	0
29	8	69	0	109	8	149	8	189	0
30	0	70	0	110	0	150	0	190	0
31	0	71	0	111	0	151	0	191	0
32	4	72	4	112	0	152	0	192	0
33	0	73	8	113	8	153	8	193	8
34	8	74	8	114	0	154	0	194	8
35	0	75	0	115	0	155	0	195	0
36	4	76	0	116	8	156	0	196	4
37	8	77	0	117	8	157	8	197	8
38	0	78	0	118	0	158	0	198	0
39	0	79	0	119	0	159	0	199	0
40	8	80	8	120	0	160	8	200	12

2 Sum of Four Squares

For the sum of two squares in the previous section, we rewrote Cauchy's second ${}_1\psi_1$ identity. Now, for the sum of four squares, we start with the Cauchy's first ${}_1\psi_1$ identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{(1-aq^n)} x^n = \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty^2}{(x;q)_\infty \left(\frac{q}{x};q\right)_\infty (a;q)_\infty \left(\frac{q}{a};q\right)_\infty}$$

. Set $a = -1$, and rewrite as explained by Johnson in Section 7.3. The result is $1+8\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k} - 8\sum_{k=1}^{\infty}\frac{4kq^{4k}}{1-q^{4k}} = \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4$, which tells us that the coefficient of q^m on the left side is the number of ways of writing m as a sum of four squares. Denoting the number of ways of writing m as a sum of four squares with $S_4(m)$, we have

$$\sum_{m=0}^{\infty} S_4(m) q^m = 1+8\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k} - 8\sum_{k=1}^{\infty}\frac{4kq^{4k}}{1-q^{4k}}.$$

In the next two subsections, we rewrite the two sums on the right side in terms of the sum of the divisors of *m*.

2.1 First Sum

In the article Divisors, we saw Lambert's theorem $\sum_{k=1}^{\infty}\frac{q^k}{1-q^k} = \sum_{m=1}^{\infty} d(m) q^m$, where $d(n)$ denotes the <u>number</u> of divisors for *m*. Another version of Lambert's theorem is $\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k} = \sum_{m=1}^{\infty}\sigma(m) q^m$, where $\sigma(m)$ denotes the <u>sum</u> of the divisors of *m*. Substituting into $8\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}$ results in $8\sum_{m=1}^{\infty}\sigma(m) q^m$, which tells us that the coefficient of q^m in the first sum is 8 times the sum of the divisors of *m*.

2.2 Second Sum

For the second sum in the four-squares expression, take Lambert's theorem and replace q with q^{4k} and multiply by 4 to produce $\sum_{k=1}^{\infty}\frac{4kq^{4k}}{1-q^{4k}} = \sum_{m=1}^{\infty} 4\sigma(m) q^{4m}$. Substituting into the second sum results in $8\sum_{m=1}^{\infty} 4\sigma(m) q^{4m}$. In words, the coefficient of q^m in the second sum is 8 times the sum of the divisors of *m* that are multiples of 4.

2.3 Examples

The result is $\sum_{m=0}^{\infty} S_4(m)q^m = 1 + 8\sum_{m=1}^{\infty}\sigma(m)q^m - 8\sum_{m=1}^{\infty}4\sigma(m)q^{4m}$, which tells us that the coefficient of q^m is 8 times the sum of the divisors of *m* that are not multiples of 4. This is quite an interesting result, and is called the Jacobi four square theorem.

The first 25 terms of the resulting polynomial are: 248*q**25 + 96*q**24 + 192*q**23 + 288*q**22 + 256*q**21 + 144*q**20 + 160*q**19 + 312*q**18 + 144*q**17 + 24*q**16 + 192*q**15 + 192*q**14 + 112*q**13 + 96*q**12 + 96*q**11 + 144*q**10 + 104*q**9 + 24*q**8 + 64*q**7 + 96*q**6 + 48*q**5 + 24*q**4 + 32*q**3 + 24*q**2 + 8*q + 1.

This polynomial demonstrates that there are 24 ways to write 4 as a sum of four squares. There are $2^4 = 16$ permutations of -1 and 1, plus 4 permutations of 2 and 0, plus 4 permutations of -2 and 0.

For *m* up to 100, Table 2 shows the coefficients that represent the number of ways to write *m* as a sum of four squares.

Table 2: Number of ways to write *m* as a sum of four squares

m	$S_4(m)$	*m*	$S_4(m)$	*m*	$S_4(m)$	*m*	$S_4(m)$	*m*	$S_4(m)$
1	8	21	256	41	336	61	496	81	968
2	24	22	288	42	768	62	768	82	1008
3	32	23	192	43	352	63	832	83	672
4	24	24	96	44	288	64	24	84	768
5	48	25	248	45	624	65	672	85	864
6	96	26	336	46	576	66	1152	86	1056
7	64	27	320	47	384	67	544	87	960
8	24	28	192	48	96	68	432	88	288
9	104	29	240	49	456	69	768	89	720
10	144	30	576	50	744	70	1152	90	1872
11	96	31	256	51	576	71	576	91	896
12	96	32	24	52	336	72	312	92	576
13	112	33	384	53	432	73	592	93	1024
14	192	34	432	54	960	74	912	94	1152
15	192	35	384	55	576	75	992	95	960
16	24	36	312	56	192	76	480	96	96
17	144	37	304	57	640	77	768	97	784
18	312	38	480	58	720	78	1344	98	1368
19	160	39	448	59	480	79	640	99	1248
20	144	40	144	60	576	80	144	100	744

