Sums of Squares

Introduction

This topic introduces a special type of q-hypergeometric series, the ${}_1\psi_1$ function, formulated by Ramanujan. In general, the "psi" functions are distinguished from the "phi" functions in the previous topic by having the same number of terms in the numerator as in the denominator; in other words, the notation is ${}_r\psi_r$ instead of ${}_{r+1}\phi_r$. From this Ramanujan ${}_1\psi_1$ function we derive the Cauchy ${}_1\psi_1$ function and a special case of the Cauchy ${}_1\psi_1$ function.

We then use the special case of the Cauchy ${}_1\psi_1$ function to derive a polynomial for which the coefficient of q^m is the number of ways of writing *m* as a sum of two squares. In a similar way, we use the Cauchy ${}_1\psi_1$ function to derive a polynomial for which the coefficient of q^m is the number of ways of writing *m* as a sum of four squares.

See Chapters 6 and 7 of *An Introduction to q-analysis* by Warren P. Johnson.

Section I: Ramanujan's ${}_1\psi_1$ Formula

In this and future topics we will be using a special type of q-hypergeometric series which uses the Greek letter psi instead of phi. For example, our simple q-hypergeometric series in the previous topic was ${}_2\phi_1$, and recall that ${}_2\phi_1 = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} x^n$. The function we are using in this topic is ${}_1\psi_1$, where ${}_1\psi_1 = \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n$. This demonstrates three characteristics of a psi q-hypergeometric series that distinguishes it from a phi q-hypergeometric series:

- There is no $(q;q)_n$ term in the denominator.
- The number of numerator terms is the same as the number of denominator terms.
- The summation is over both negative and positive integers.

RamanujanOnePsiOne(inf): Ramanujan discovered a sum and product identity for ${}_1\psi_1$ in terms of eight q-shifted factorials:

$${}_1\psi_1 = \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty \left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty \left(\frac{b}{ax};q\right)_\infty (b;q)_\infty \left(\frac{q}{a};q\right)_\infty}.$$

Johnson explains at least four proofs of Ramanujan's identity in Section 6.1 of *An Introduction to q-analysis.*

For convergence, we require $|q|<1$ and $\left|\frac{b}{a}\right|<|x|<1$. In addition to these constraints, we must avoid an infinite number of parameter values that make a q-shifted factorial in the denominator zero. For example, we must avoid $a = q$, $a=q^2$, … , $b = ax$, $b = 1$, $b = 1/q$, $b = 1/q^2$, … .

To study convergence, I used the Python function **ExperimentRamanujanOnePsiOne** over a range of feasible values for *a*, c, *x*, and *q*. The results in the table below show increasing accuracy as the value of *inf* increases. The column labels are values for *q* and the row labels are values for *a* within values for *x*. The values for b are not shown, but were calculated using *a***x* - 0.01.

We see that the slowest convergence is when there is a large value of *q* in combination with small values of the other parameters.

qqnNeg(A, n, inf): This utility function, called by **RamanujanOnePsiOne**, implements the q-shifted factorial adapted for negative n as derived in Section 6.1 of *An Introduction to q-Analysis* by Warren P. Johnson.

Since $(a;q)_\infty = (a;q)_n (aq^n;q)_\infty$, we have $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$.

RamanujanOnePsiOne

(rows: a within x; columns: q)

inf = 20

Average of PerDiff	Column Labels						
Row Labels	0.13	0.26	0.39	0.52	0.65	0.78	0.91
0.14							
0.15	0.00	0.00	0.00	0.00	0.00	0.00	1,230.14
0.30	0.00	0.00	0.00	0.02	0.01	0.68	7.72
0.45	0.01	0.06	0.06	0.06	0.04	3.06	0.69
0.60	0.04	0.01	0.03	0.03	0.13	10.17	0.99
0.75	0.47	0.38	0.46	0.50	0.84	2.49	1.00
0.90	0.99	0.40	0.53	0.68	0.28	1.32	1.00
0.28							
0.15	0.00	0.00	0.00	0.01	0.00	0.16	34.06
0.30	0.03	0.01	0.02	0.02	0.09	6.96	0.03
0.45	0.95	0.38	0.50	0.63	0.24	1.12	1.00
0.60	0.30	0.41	0.21	0.35	0.94	1.54	1.00
0.75	0.34	0.47	0.37	0.48	0.45	1.24	1.00
0.90	0.36	1.00	0.42	0.75	0.79	0.83	0.99
0.42							
0.15	0.01	0.04	0.04	0.04	0.01	0.77	7.69
0.30	0.93	0.37	0.48	0.60	0.23	0.98	0.99
0.45	0.32	0.42	0.32	0.40	0.08	0.43	1.00
0.60	0.34	0.95	0.40	0.70	0.72	0.66	0.99
0.75	0.36	0.28	0.46	0.31	0.36	0.51	1.08
0.90	0.64	0.62	1.12	0.69	0.84	0.10	1.05
0.56							
0.15	0.03	0.00	0.02	0.01	0.06	3.18	3.28
0.30	0.29	0.38	0.20	0.32	0.79	1.21	0.92
0.45	0.33	0.91	0.38	0.66	0.66	0.55	0.99
0.60	0.67	0.64	0.77	0.67	0.73	0.92	0.98
0.75	0.60	0.58	0.42	0.67	3.41	0.80	0.79
0.90	0.56	0.55	0.52	1.11	0.59	0.48	0.68
0.70							
0.15	0.41	0.32	0.38	0.40	0.60	2.07	0.87
0.30	0.30	0.41	0.32	0.40	0.34	0.77	1.04
0.45	0.32	0.26	0.41	0.26	0.30	0.32	1.23
0.60	0.58	0.57	0.40	0.65	3.22	0.72	1.09
0.75	0.53	0.52	0.50	1.11	0.57	0.59	1.49
0.90	0.72	0.71	0.71	0.67	1.11	0.46	1.50
0.84							
0.15	0.73	0.30	0.38	0.46	0.17	0.58	0.84
0.30	0.28	0.73	0.31	0.52	0.50	0.34	0.86
0.45	0.58	0.56	0.97	0.60	0.70	0.00	1.28
0.60	0.50	0.49	0.46	0.94	0.49	0.34	1.54
0.75	0.68	0.68	0.67	0.64	1.02	0.41	0.23
0.90	0.84	0.84	0.83	0.83	0.81	1.06	0.40
Grand Total	**0.42**	**0.42**	**0.39**	**0.48**	**0.61**	**1.33**	**36.46**

inf = 100

Average of PerDiff	Column Labels						
Row Labels	0.13	0.26	0.39	0.52	0.65	0.78	0.91
0.14							
0.15	0.00	0.00	0.00	0.00	0.00	0.00	41.24
0.30	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.45	0.00	0.00	0.00	0.00	0.00	0.00	0.03
0.60	0.00	0.00	0.00	0.00	0.00	0.00	0.05
0.75	0.01	0.01	0.01	0.01	0.02	0.08	1.17
0.90	0.02	0.01	0.01	0.01	0.01	0.03	3.64
0.28							
0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.01
0.30	0.00	0.00	0.00	0.00	0.00	0.00	0.01
0.45	0.02	0.01	0.01	0.01	0.00	0.02	2.20
0.60	0.01	0.01	0.00	0.01	0.02	0.03	0.88
0.75	0.01	0.01	0.01	0.01	0.01	0.02	0.01
0.90	0.01	0.02	0.01	0.02	0.02	0.01	0.07
0.42							
0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.30	0.02	0.01	0.01	0.01	0.00	0.02	1.46
0.45	0.01	0.01	0.01	0.01	0.00	0.01	0.09
0.60	0.01	0.02	0.01	0.01	0.01	0.01	0.05
0.75	0.01	0.01	0.01	0.01	0.01	0.01	0.17
0.90	0.12	0.11	0.20	0.12	0.15	0.00	0.52
0.56							
0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.30	0.01	0.01	0.00	0.01	0.02	0.02	0.39
0.45	0.01	0.02	0.01	0.01	0.01	0.01	0.04
0.60	0.16	0.15	0.18	0.16	0.17	0.21	0.30
0.75	0.09	0.09	0.06	0.10	0.50	0.11	0.54
0.90	0.06	0.06	0.05	0.12	0.06	0.05	0.35
0.70							
0.15	0.01	0.01	0.01	0.01	0.01	0.04	0.22
0.30	0.01	0.01	0.01	0.01	0.01	0.02	0.01
0.45	0.01	0.01	0.01	0.01	0.01	0.01	0.07
0.60	0.08	0.08	0.06	0.09	0.47	0.10	0.42
0.75	0.05	0.05	0.05	0.11	0.06	0.06	0.16
0.90	0.20	0.20	0.20	0.19	0.31	0.13	0.27
0.84							
0.15	0.02	0.01	0.01	0.01	0.00	0.01	0.36
0.30	0.01	0.02	0.01	0.01	0.01	0.01	0.01
0.45	0.10	0.10	0.18	0.11	0.13	0.00	0.25
0.60	0.05	0.05	0.05	0.10	0.05	0.04	0.17
0.75	0.19	0.19	0.19	0.18	0.29	0.11	0.21
0.90	0.44	0.44	0.44	0.44	0.43	0.56	0.81
Grand Total	**0.05**	**0.05**	**0.05**	**0.05**	**0.08**	**0.05**	**1.56**

CauchyOnePsiOne(inf): As explained in Section 7.1 of Johnson's book, about 80 years before Ramanujan, Cauchy developed an identity that is a special case of Ramanujan's identity when b is replaced with aq. Starting with the denominator $(b;q)_n$ in the summation, we have $(b;q)_n$ = $(aq;q)_n$ and we can expand this q-shifted factorial as $(aq;q)_n$ = $\frac{1-a}{1-a}(1-aq)(1-aq^2)\cdots(1-aq^{n-1})(1-aq^n) = \frac{(a;q)_n}{1-a}(1-aq^n)$. This results in the summation term $(1-a)\sum_{n=-\infty}^{\infty}\frac{1}{(1-aq^n)}x^n$. Now, replacing replace b with aq in the product term, results in the product term $\frac{(ax;q)_\infty\left(\frac{q}{ax};q\right)_\infty(q;q)_\infty(q;q)_\infty}{(x;q)_\infty\left(\frac{q}{x};q\right)_\infty(aq;q)_\infty\left(\frac{q}{a};q\right)_\infty}$. Dividing both sides by $(1-a)$ changes the $(aq;q)_\infty$ term to $(1-a)$ $(aq;q)_\infty = (a;q)_\infty$, and we have the Cauchy ${}_1\psi_1$ identity: $\sum_{n=-\infty}^{\infty}\frac{1}{(1-aq^n)}x^n$ = $\frac{(ax;q)_\infty\left(\frac{q}{ax};q\right)_\infty(q;q)_\infty^2}{(x;q)_\infty\left(\frac{q}{x};q\right)_\infty(a;q)_\infty\left(\frac{q}{a};q\right)_\infty}$. We use the Cauchy ${}_1\psi_1$ identity for sums of four squares in Section III.

The constraints for convergence are $|q|<|x|<1$ and $|q|<|a|<1$. To avoid a zero denominator, we must also avoid values $a = q, 1/q, 1/q^2, \ldots, q^2, q^3\ldots$, and $x = q, 1/q, 1/q^2, \ldots, q^2, q^3\ldots$. To achieve these constraints, I use ranges for x and a but calculate $q = x \times a$.

To study convergence, I used the Python function **ExperimentCauchyOnePsiOne** over a range of feasible values for a, x, and q. The results in the table below show rapid convergence as the value of *inf* increases. The row labels are values for x, and the columns labels are values for a. The calculated values for q range from 0.02 to 0.16, and the results are averaged over these values.

CauchyOnePsiOne

(rows: x; columns: a, averaged over q = {0.02, …, 0.16}

inf = 20

Average of Diff	Column Labels						
Row Labels	**0.14**	**0.28**	**0.42**	**0.56**	**0.70**	**0.84**	**Grand Total**
0.14	0.0000	0.0000	0.0000	0.0000	0.0043	0.3273	0.0553
0.28	0.0000	0.0000	0.0000	0.0000	0.0043	0.3273	0.0553
0.42	0.0000	0.0000	0.0000	0.0000	0.0019	0.1553	0.0262
0.56	0.0000	0.0000	0.0000	0.0000	0.0023	0.1870	0.0316
0.70	0.0019	0.0019	0.0019	0.0018	0.0008	0.2072	0.0359
0.84	0.1606	0.1606	0.1606	0.1606	0.1577	0.0642	0.1440
Grand Total	**0.0271**	**0.0271**	**0.0271**	**0.0271**	**0.0286**	**0.2114**	**0.0580**

inf = 50

Average of Diff	Column Labels						
Row Labels	**0.14**	**0.28**	**0.42**	**0.56**	**0.70**	**0.84**	**Grand Total**
0.14	0.0000	0.0000	0.0000	0.0000	0.0000	0.0032	0.0005
0.28	0.0000	0.0000	0.0000	0.0000	0.0000	0.0032	0.0005
0.42	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007	0.0001
0.56	0.0000	0.0000	0.0000	0.0000	0.0000	0.0010	0.0002
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0012	0.0002
0.84	0.0009	0.0009	0.0009	0.0009	0.0009	0.0006	0.0008
Grand Total	**0.0001**	**0.0001**	**0.0001**	**0.0001**	**0.0001**	**0.0016**	**0.0004**

CauchyOnePsiOneCase(inf): For this special case derived by Cauchy, replace q with q^2 and x with xq to obtain $\sum_{n=-\infty}^{\infty} \frac{x^n q^{2n}}{1+q^{2n}} = \frac{(axq;q^2)_\infty \left(\frac{q}{ax};q^2\right)_\infty (q^2;q^2)_\infty^2}{(xq;q^2)_\infty \left(\frac{q}{x};q^2\right)_\infty (a;q^2)_\infty \left(\frac{q^2}{a};q^2\right)_\infty}$. Then set a = -1 to obtain

$\sum_{n=-\infty}^{\infty} \frac{x^n q^{2n}}{1+q^{2n}} = \frac{(-xq;q^2)_\infty \left(-\frac{q}{x};q^2\right)_\infty (q^2;q^2)_\infty^2}{(xq;q^2)_\infty \left(\frac{q}{x};q^2\right)_\infty (-1;q^2)_\infty \left(-q^2;q^2\right)_\infty}$. Note that $(-1;q^2)_\infty = 2(1+q^2)(1+q^4)\cdots$ $= 2\left(-q^2;q^2\right)$, resulting in the special case of the Cauchy ${}_1\psi_1$ identity $\sum_{n=-\infty}^{\infty} \frac{2x^n q^{2n}}{1+q^{2n}} =$

$$\frac{(-xq;q^2)_\infty \left(-\frac{q}{x};q^2\right)_\infty (q^2;q^2)_\infty^2}{(xq;q^2)_\infty \left(\frac{q}{x};q^2\right)_\infty \left(-q^2;q^2\right)_\infty^2}.$$

We will use the special case of the Cauchy ${}_1\psi_1$ identity in Section II for sums of two squares.

Again requiring $|q| < |x| < 1$, or equivalently $|q| < |x| < \frac{1}{q}$, and avoiding problem values for x such as $x = q$, q^3, q^5, … , $1/q$, $1/q^3$, $1/q^5$, … , the identity quickly converges.

To study convergence, I used the Python function **ExperimentCauchyOnePsiOneCase** over a range of feasible values for x and q. As with the previous results, as *inf* is increased from 20 to 50 to 100, the average percent difference decreases from 11% to 0.5%, to 0.0035%.

Section II: Sum of Two Squares

SumOfTwoSquares(inf): From the previous section, take the special case of the Cauchy ${}_1\psi_1$ identity $\sum_{n=-\infty}^{\infty} \frac{2x^n q^{2n}}{1+q^{2n}} = \frac{(-xq;q^2)_\infty \left(-\frac{q}{x};q^2\right)_\infty (q^2;q^2)_\infty^2}{(xq;q^2)_\infty \left(\frac{q}{x};q^2\right)_\infty \left(-q^2;q^2\right)_\infty^2}$ and set x = 1 to produce $\sum_{n=-\infty}^{\infty} \frac{2q^{2n}}{1+q^{2n}} =$ $\frac{(-q;q^2)_\infty \left(-q;q^2\right)_\infty (q^2;q^2)_\infty^2}{(q;q^2)_\infty \left(q;q^2\right)_\infty \left(-q^2;q^2\right)_\infty^2}$. The right side can be written $\left(\frac{\left(-q;q^2\right)_\infty \left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty \left(-q^2;q^2\right)_\infty}\right)^2$, which tells us that $\sum_{n=-\infty}^{\infty} \frac{2q^{2n}}{1+q^{2n}}$ is a perfect square. This product term can be further rewritten using Euler's odd equals distinct theorem $(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}$ from the Partition Theory topic and Jacobi's triple identity (with z = 1) $(-q;q^2)_\infty \left(-q;q^2\right)_\infty (q^2;q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2}$ from the q-Identities topic. This results in the considerably simplified expression

$2\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1+q^{2n}} = \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2$. Since the right side equals $\left(\sum_{j=-\infty}^{\infty} q^{j^2}\right)\left(\sum_{k=-\infty}^{\infty} q^{k^2}\right) = \sum_{j,k=-\infty}^{\infty} q^{j^2+k^2}$, we have $2\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1+q^{2n}} = \sum_{j,k=-\infty}^{\infty} q^{j^2+k^2}$, which tells us that the coefficient of q^m on the left side is the number of ways of writing *m* as a sum of two squares. Because of symmetry, we can simplify $\sum_{n=-\infty}^{\infty} \frac{2q^{2n}}{1+q^{2n}} = 1+4\sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}}$ and conclude that $\sum_{m=0}^{\infty} S_2(m)q^m = 1+4\sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}}$, where $S_2(m)$ is the number of ways of writing *m* as a sum of two squares. It is important to note that this allows for both positive and negative numbers and permutations.

The first 25 terms of the resulting polynomial are: 12*q**25 + 8*q**20 + 4*q**18 + 8*q**17 + 4*q**16 + 8*q**13 + 8*q**10 + 4*q**9 + 4*q**8 + 8*q**5 + 4*q**4 + 4*q**2 + 4*q + 1.

This polynomial demonstrates that there are four ways to write 1, 2, and 4 as a sum of two squares. Checking this, we find that for 1 it is 1^2+0^2, 0^2+1^2, $(-1)^2+0^2$, $0^2+(-1)^2$; for 2 it is 1^2+1^2, 1^2+1^2, $1^2+(-1)^2$, $(-1)^2+1^2$; and for 4 it is 2^2+0^2, 0^2+2^2, $(-2)^2+0^2$, $0^2+(-2)^2$.

For *m* up to 200, the table below shows the coefficients that represent the number of ways to write *m* as a sum of two squares. In this range, $S_2(m)$ ranges from 0 to 16, and for most *m* we cannot write m as a sum of two squares ($S_2(m) = 0$).

Number of ways to write *m* as a sum of two squares									
m	$S_2(m)$	m	$S_2(m)$	m	$S_2(m)$	m	$S_2(m)$	m	$S_2(m)$
1	4	41	8	81	4	121	4	161	0
2	4	42	0	82	8	122	8	162	4
3	0	43	0	83	0	123	0	163	0
4	4	44	0	84	0	124	0	164	8
5	8	45	8	85	16	125	16	165	0
6	0	46	0	86	0	126	0	166	0
7	0	47	0	87	0	127	0	167	0
8	4	48	0	88	0	128	4	168	0
9	4	49	4	89	8	129	0	169	12
10	8	50	12	90	8	130	16	170	16
11	0	51	0	91	0	131	0	171	0
12	0	52	8	92	0	132	0	172	0
13	8	53	8	93	0	133	0	173	8
14	0	54	0	94	0	134	0	174	0
15	0	55	0	95	0	135	0	175	0
16	4	56	0	96	0	136	8	176	0
17	8	57	0	97	8	137	8	177	0
18	4	58	8	98	4	138	0	178	8
19	0	59	0	99	0	139	0	179	0
20	8	60	0	100	12	140	0	180	8
21	0	61	8	101	8	141	0	181	8
22	0	62	0	102	0	142	0	182	0
23	0	63	0	103	0	143	0	183	0
24	0	64	4	104	8	144	4	184	0
25	12	65	16	105	0	145	16	185	16
26	8	66	0	106	8	146	8	186	0
27	0	67	0	107	0	147	0	187	0
28	0	68	8	108	0	148	8	188	0
29	8	69	0	109	8	149	8	189	0
30	0	70	0	110	0	150	0	190	0
31	0	71	0	111	0	151	0	191	0
32	4	72	4	112	0	152	0	192	0
33	0	73	8	113	8	153	8	193	8
34	8	74	8	114	0	154	0	194	8
35	0	75	0	115	0	155	0	195	0
36	4	76	0	116	8	156	0	196	4
37	8	77	0	117	8	157	8	197	8

38	0	78	0	118	0	158	0	198	0
39	0	79	0	119	0	159	0	199	0
40	8	80	8	120	0	160	8	200	12

JacobiTwoSquare(n): The right side of the previous expression $1+4\sum_{n=1}^{\infty}\frac{q^{2n}}{1+q^{2n}}$ can be written $1+4\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(q^{n(4k+1)}-q^{n(4k+3)}\right)$, and simplified to $1+4\sum_{n=1}^{\infty}\left(d_{1,4}(m)-d_{3,4}(m)\right)q^m$, where $d_{1,4}(m)$ is the number of divisors of *m* that are congruent to 1 mod 4 and $d_{3,4}(m)$ is the number of divisors of *m* that are congruent to 3 mod 4 (see Section 7.2 of Johnson). Now we have $\sum_{m=0}^{\infty}S_2(m)q^m =$

$$1+4\sum_{n=1}^{\infty}\left(d_{1,4}(m)-d_{3,4}(m)\right)q^m .$$

However, note that in this expression, we no longer need the polynomials and can simply write $S_2(m) = 4\left(d_{1,4}(m)-d_{3,4}(m)\right)$, which is known as Jacobi's two square theorem. This function calculates $S_2(m)$ for m = 0 to *n*.

DivisorMod4Count(r, n): This utility function, called by **JacobiTwoSquare**, counts the number of divisors of n that are congruent to r mod 4.

Section III: Sum of Four Squares

JacobiFourSquare(n): For the sum of two squares in the previous section, we rewrote the special case of the Cauchy ${}_1\psi_1$ identity. Here, we start with the main Cauchy ${}_1\psi_1$ identity $\sum_{n=-\infty}^{\infty}\frac{1}{(1-aq^n)}x^n = \frac{(ax;q)_\infty\left(\frac{q}{ax};q\right)_\infty (q;q)_\infty^2}{(x;q)_\infty\left(\frac{q}{x};q\right)_\infty (a;q)_\infty\left(\frac{q}{a};q\right)_\infty}$, set a = -1, and rewrite as explained by Johnson in Section 7.3. The result is $1+8\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-8\sum_{k=1}^{\infty}\frac{4kq^{4k}}{1-q^{4k}} = \left(\sum_{n=-\infty}^{\infty}q^{n^2}\right)^4$, which tells us that the coefficient of q^m on the left side is the number of ways of writing *m* as a sum of four squares. Denoting the number of ways of writing *m* as a sum of four squares with $S_4(m)$, we have

$\sum_{m=0}^{\infty}S_4(m)q^m = 1+8\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-8\sum_{k=1}^{\infty}\frac{4kq^{4k}}{1-q^{4k}}$. Now we can express sums on the right side in terms of the sum of the divisors of m from the Partition Theory topic.

First Sum: In Section 3 of the Partition Geometry topic, Lambert's theorem proved that $\sum_{k=1}^{\infty}\frac{q^k}{1-q^k} = \sum_{m=1}^{\infty}d(m)q^m$, where $d(n)$ denotes the number of divisors for *m*. By multiplying the left terms by *k*, we have $\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k} = \sum_{m=1}^{\infty}\sigma(m)q^m$, where $\sigma(m)$ denotes the sum of the divisors of *m*. Substituting into the left sum results in $8\sum_{m=1}^{\infty}\sigma(m)q^m$. In words, the coefficient of q^m in the first sum is 8 times the sum of the divisors of *m*.

Second Sum: For the second sum, take Lambert's theorem and replace q with q^{4k} and multiply by 4 to produce $\sum_{k=1}^{\infty}\frac{4kq^{4k}}{1-q^{4k}} = \sum_{m=1}^{\infty}4\sigma(m)q^{4m}$. Substituting into the second sum results in $8\sum_{m=1}^{\infty}4\sigma(m)q^{4m}$. In words, the coefficient of q^m in the second sum is 8 times the sum of the divisors of *m* that are multiples of 4.

The result is $\sum_{m=0}^{\infty}S_4(m)q^m = 1+8\sum_{m=1}^{\infty}\sigma(m)q^m - 8\sum_{m=1}^{\infty}4\sigma(m)q^{4m}$, which tells us that the coefficient of q^m is 8 times the sum of the divisors of *m* that are not multiples of 4. This is quite an interesting result, and is called the Jacobi four square theorem.

The first 25 terms of the resulting polynomial are: 248*q**25 + 96*q**24 + 192*q**23 + 288*q**22 + 256*q**21 + 144*q**20 + 160*q**19 + 312*q**18 + 144*q**17 + 24*q**16 + 192*q**15 + 192*q**14 + 112*q**13 + 96*q**12 + 96*q**11 + 144*q**10 + 104*q**9 + 24*q**8 + 64*q**7 + 96*q**6 + 48*q**5 + 24*q**4 + 32*q**3 + 24*q**2 + 8*q + 1.

This polynomial demonstrates that there are 24 ways to write 4 as a sum of four squares. There are $2^4 = 16$ permutations of -1 and 1, plus 4 permutations of 2 and 0, plus 4 permutations of -2 and 0.

For *m* up to 100, the table below shows the coefficients that represent the number of ways to write *m* as a sum of four squares.

Number of ways to write *m* as a sum of four squares									
m	$S_4(m)$	m	$S_4(m)$	m	$S_4(m)$	m	$S_4(m)$	m	$S_4(m)$
1	8	21	256	41	336	61	496	81	968
2	24	22	288	42	768	62	768	82	1008
3	32	23	192	43	352	63	832	83	672
4	24	24	96	44	288	64	24	84	768
5	48	25	248	45	624	65	672	85	864
6	96	26	336	46	576	66	1152	86	1056
7	64	27	320	47	384	67	544	87	960
8	24	28	192	48	96	68	432	88	288
9	104	29	240	49	456	69	768	89	720
10	144	30	576	50	744	70	1152	90	1872
11	96	31	256	51	576	71	576	91	896
12	96	32	24	52	336	72	312	92	576
13	112	33	384	53	432	73	592	93	1024
14	192	34	432	54	960	74	912	94	1152
15	192	35	384	55	576	75	992	95	960
16	24	36	312	56	192	76	480	96	96
17	144	37	304	57	640	77	768	97	784
18	312	38	480	58	720	78	1344	98	1368
19	160	39	448	59	480	79	640	99	1248
20	144	40	144	60	576	80	144	100	744

