Partitions with Multiple Restrictions

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Recall that partition-generating functions produce a q-polynomial in which the coefficient of q^n is the number of partitions for the integer n. In the previous article (Partitions with Distinct and Repeated Parts), we demonstrated seven generating functions for partitions with distinct, repeated, odd, and/or even parts and two generating functions that applied single restrictions related to the number of parts. In this article, the generating functions apply two restrictions on the parts.

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapter 3 of *An Introduction to q-analysis* by Warren P. Johnson.

1 Generating Functions Overview

These partition-generating functions consist of more than the infinite q-shifted factorial. Cayley's theorem is different from the other generating functions because it uses a q-binomial coefficient instead of a q-shifted factorial. The other generating functions contain finite q-shifted factorials of the form $(q;q)_n$ or $\frac{1}{(q;q)_n}$ in addition to, or instead of, infinite q-shifted factorials.

The generating functions are listed in Table 1.

Table 1: Partition-Generating Functions

#	Generating Function	Type of Partition
1	$\begin{pmatrix} i+j \\ i \end{pmatrix}_q$	at most i parts, each part at most j (Cayley's theorem)
2	$\dfrac{q^{2k}}{(q;q^2)_\infty (q^2;q^2)_k}$	k is largest repeated part, or k even parts (Andrews-Deutsch theorem)
3	$\dfrac{q^{dk}}{(q^d;q^d)_k} \dfrac{(q^d;q^d)_\infty}{(q;q)_\infty}$	k is the largest part that occurs at least d times (Smoot-Yang theorem)
4	$\dfrac{(-q;q^2)_n}{(q^2;q^2)_n}$	at most n parts for which even parts may be repeated but odd parts are distinct (ee partition)
5	$q^n \dfrac{(-q;q^2)_n}{(q^2;q^2)_n}$	exactly n parts for which odd parts may be repeated but even parts are distinct (oo partition)

Table 2 shows the partitions for n = 4 to 9 to be used to demonstrate the theorems.

Table 2: Partition Examples	
n	**Partitions**
4	(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)
5	(5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1)
6	(6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1), (2 1 1 1 1), (1 1 1 1 1 1)
7	(7), (6 1), (5 2), (5 1 1), (4 3), (4 2 1), (4 1 1 1), (3 3 1), (3 2 2), (3 2 1 1), (3 1 1 1 1), (2 2 2 1), (2 2 1 1 1), (2 1 1 1 1 1), (1 1 1 1 1 1 1)
8	(8), (7 1), (6 2), (6 1 1), (5 3), (5 2 1), (5 1 1 1), (4 4), (4 3 1), (4 2 2), (4 2 1 1), (4 1 1 1 1), (3 3 2), (3 3 1 1), (3 2 2 1), (3 2 1 1 1), (3 1 1 1 1 1), (2 2 2 2), (2 2 2 1 1), (2 2 1 1 1 1), (2 1 1 1 1 1 1), (1 1 1 1 1 1 1 1)
9	(9), (8 1), (7 2), (7 1 1), (6 3), (6 2 1), (6 1 1 1), (5 4), (5 3 1), (5 2 2), (5 2 1 1), (5 1 1 1 1), (4 4 1), (4 3 2), (4 3 1 1), (4 2 2 1), (4 2 1 1 1), (4 1 1 1 1 1), (3 3 3), (3 3 2 1), (3 3 1 1 1), (3 2 2 2), (3 2 2 1 1), (3 2 1 1 1 1), (3 1 1 1 1 1 1), (2 2 2 2 1), (2 2 2 1 1 1), (2 2 1 1 1 1 1), (2 1 1 1 1 1 1 1), (1 1 1 1 1 1 1 1 1)

2 At Most i Parts, Each Part at Most j (Cayley)

This theorem of Cayley, demonstrates that $\binom{i+j}{i}_q$ is the generating function for partitions with at most i parts, each part at most j.

(i, j) = (3, 2): The polynomial is q**6 + q**5 + 2*q**4 + 2*q**3 + 2*q**2 + q + 1.

Consider n = 4. The polynomial predicts that there are two partitions of 4 into at most three parts, each part at most 2. Table 2 confirms that the two partitions are (2, 2) and (2, 1, 1).

(i, j) = (3, 3): The polynomial is q**9 + q**8 + 2*q**7 + 3*q**6 + 3*q**5 + 3*q**4 + 3*q**3 + 2*q**2 + q + 1.

For this example, consider n = 6. The polynomial predicts that there are three partitions of 6 into at most three parts, each part at most 3. Table 2 confirms that the three partitions are (3 3), (3 2 1), and (2 2 2).

(i, j) = (4, 2): The polynomial is q**8 + q**7 + 2*q**6 + 2*q**5 + 3*q**4 + 2*q**3 + 2*q**2 + q + 1.

Consider n = 5. The polynomial predicts that there are two partitions of 5 into at most four parts, each part at most 2, and Table 2 confirms that the two partitions are (2 2 1) and (2 1 1 1).

3 Largest repeated part k, or k even parts (Andrews-Deutsch)

The Andrews-Deutsch theorem proves that the function $\frac{q^{2k}}{(q;q^2)_\infty (q^2;q^2)_k}$ generates partitions for

a) the largest repeated part is k, and also for partitions with

b) k even parts.

k = 2: The lower-order terms of the generated polynomial are 7*q**9 + 5*q**8 + 3*q**7 + 2*q**6 + q**5 + q**4. The coefficients indicate the number of partitions for a) the largest repeated part being 2, and b) 2 even parts. The partitions that satisfy the restrictions for n = 4 to 9 are shown in Table 3.

Table 3: Andrews-Deutsch Results for k = 2			
n	**Number of Partitions**	**Largest Repeated Part 2**	**2 Even Parts**
4	1	(2 2)	(2 2)
5	1	(2 2 1)	(2 2 1)
6	2	(2 2 2), (2 2 1 1)	(4, 2), (2 2 1 1)
7	3	(3 2 2), (2 2 2 1), (2 2 1 1 1)	(4 2 1), (3 2 2), (2 2 1 1 1)
8	5	(4 2 2), (3 2 2 1), (2 2 2 2), (2 2 2 1 1), (2 2 1 1 1 1)	(6 2), (4 4), (4 2 1 1), (3 2 2 1), (2 2 1 1 1 1),
9	7	(5 2 2), (4 2 2 1), (3 2 2 2), (3 2 2 1 1), (2 2 2 2 1), (2 2 2 1 1 1), (2 2 1 1 1 1 1)	(6 2 1), (5 2 2), (4 4 1), (4 3 2), (4 2 1 1 1), (3 2 2 1 1), (2 2 1 1 1 1 1)

k = 3: The lower-order terms of the generatedpolynomial are 5*q**10 + 3*q**9 + 2*q**8 + q**7 + q**6. The coefficients indicate the number of partitions for a) the largest repeated part being 3, and b) 3 even parts. The partitions that satisfy the restrictions for n = 6 to 9 are shown in Table 4.

Table 4: Andrews-Deutsch Results for k = 3			
n	**Number of Partitions**	**Largest Repeated Part 3**	**3 Even Parts**
6	1	(3 3)	(2 2 2)
7	1	(3 3 1)	(2 2 2 1)
8	2	(3 3 2), (3 3 1 1)	(4 2 2), (2 2 2 1 1)
9	3	(3 3 3), (23 3 2 1), (3 3 1 1 1)	(4 2 2 1), (3 2 2 2), (2 2 2 1 1 1)

4 k is largest part occurring at least d times (Smoot-Yang)

The Smoot-Yang theorem states that $\frac{q^{dk}}{(q^d;q^d)_k}\frac{(q^d;q^d)_\infty}{(q;q)_\infty}$ is the generating function for partitions:

a) with exactly k parts divisible by d, and also for partitions

b) in which k is the largest part that occurs at least d times.

The infinite component of this generating function $\frac{(q^d;q^d)_\infty}{(q;q)_\infty}$ generates partitions with parts that are not divisible by d. The finite component $\frac{q^{dk}}{(q^d;q^d)_k}$ is a generating function for the k parts divisible by d. Multiplying the two components results in the generating function for partitions with exactly k parts divisible by d.

k = 3, d = 3: The lower-order terms of the generated polynomial are q**10 + q**9.

The coefficients of the generated polynomial indicate the number of partitions: a) with exactly 3 parts divisible by 3, and b) in which 3 is the largest part that occurs at least 3 times. For the integer 9, the partition (3 3 3) satisfies a) and b).

k = 2, d = 2: The lower-order terms of the generated polynomial are 11*q**10 + 8*q**9 + 5*q**8 + 4*q**7 + 2*q**6 + q**5 + q**4.

The coefficients of the generated polynomial indicate the number of partitions: a) with 2 parts divisible by 2, and b) in which 2 is the largest part that occurs at least 2 times. The partitions are (2 2) for 4; (2 2 1) for 5; (4 2) and (2 2 1 1) for 6; (4 2 1), (3 2 2), (2 2 2 1), and (2 2 1 1 1) for 7; and so on.

5 At most n parts with repeated even but distinct odd (ee)

The function $\frac{(-q;q^2)_n}{(q^2;q^2)_n}$ generates partitions with at most n parts for which even parts may be repeated but odd parts are distinct. These are called ee partitions.

For n = 5, the low-order terms of the generated polynomial are 7*q**7 + 5*q**6 + 4*q**5 + 3*q**4 + 2*q**3 + q**2 + q + 1.

Partitions of 3: Of the three partitions (3), (2 1), (1 1 1), the first two are ee partitions, and therefore the coefficient of q**3 is 3.

Partitions of 4: Of the five partitions (4), (3 1), (2 2), (2 1 1), (1 1 1 1), only three are ee partitions: (4), (3, 1), (2, 2), and therefore the coefficient of q**4 is 3.

Partitions of 5: There are seven partitions of 5: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1). The ee partitions are (5), (4 1), (3 2), (2 2 1), and therefore the coefficient of q**5 is 4.

Partitions of 6: There are 11 partitions of 6: (6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1), (2 1 1 1 1), (1 1 1 1 1 1). The ee partitions are (6), (5 1), (4 2), (3 2 1), (2 2 2) , and therefore the coefficient of q**6 is 5.

6 Exactly n parts with repeated odd but distinct even (oo)

The function $q^n \frac{(-q;q^2)_n}{(q^2;q^2)_n}$ generates partitions with at exactly n parts for which odd parts may be repeated but repeated parts are distinct. These are called oo partitions.

For n = 3, the low-order terms of the generated polynomial are 3*q**7 + 2*q**6 + q**5 + q**4 + q**3. Certainly there are no partitions with exactly 3 parts for any integer less than 3.

Partitions of 3: Of the three partitions (1 1 1) is an oo partitions, and therefore the coefficient of q**3 is 1.

Partitions of 4: Of the five partitions only (2 1 1) is an oo partitions, and therefore the coefficient of q**4 is 1.

Partitions of 5: There are seven partitions of 5: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1). The oo partition is (3 1 1), and therefore the coefficient of q**5 is 1.

Partitions of 6: There are 11 partitions of 6: (6), (5 1), (4 2), (4 1 1), (3 3), (3 2 1), (3 1 1 1), (2 2 2), (2 2 1 1), (2 1 1 1 1), (1 1 1 1 1 1). The oo partitions are (4 1 1), (3 2 1), and therefore the coefficient of q**6 is 2.