Connection between Inversions and *q*-Factorials

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This article explains concepts for inversions and their connection to *q*-factorials.

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapter 1 of *An Introduction to q-analysis* by Warren P. Johnson.

1 *q*-Factorials

q-series began with Euler and Gauss and are the polynomials resulting from "*q*-analogues." A *q*-analogue is a version of an expression or identity involving the parameter *q* that produces the original when $q = 1$. There are *q*-analogues for standard expressions such as n, $n!$, and binomial coefficients and many more complicated expressions and identities that are explained in subsequent articles.

The *q*-analogue for an integer n is the n-1 degree polynomial $[n]_q = 1 + q + q^2 + \ldots + q^{n-1}$. For example, $[1]_q$ = 1, $[2]_q$ = q + 1, and $[3]_q$ = q**2 + q + 1. Setting q = 1 results in $[3]_1$ = 1**2 + 1 + 1 = 3.

The *q*-analogue for n factorial is $n!_q = [1]_q[2]_q \ldots [n]_q$. For example, $3!_q$ = (q + 1)*(q**2 + q + 1) = q**3 + 2*q**2 + 2*q + 1. Setting $q = 1$, yields $3!_1$ = 1**3 + 1*1**2 + 1*1 + 1 = 6 = 3!

2 Integer Permutations and Inversions

An inversion is a pair of elements in a permutation that are in decreasing order. For example, 43 and 10 are inversions. For the integer sequence {1, 2, 3}, Table 1 shows the permutations and the number of inversions for each permutation.

Table 1: Permutations and Inversions for {1, 2, 3}

Permutations	**Inversions**
(3, 2, 1)	3 since 3>2, 3>1, and 2>1.
(3, 1, 2)	2 since 3>1 and 3>2.
(2, 3, 1)	2 since 2>1 and 3>1.
(1, 3, 2)	1 since 3>2.
(2, 1, 3)	1 since 2>1.
(1, 2, 3)	0

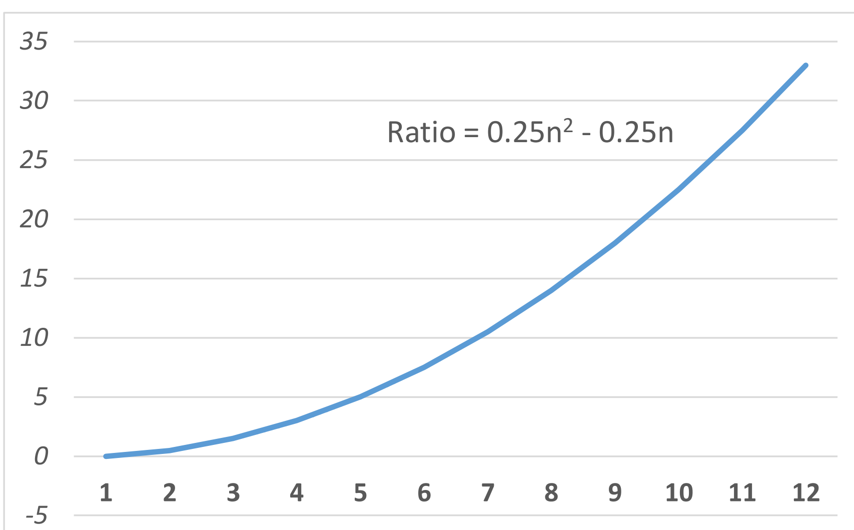
There is 1 permutation with three inversions, 2 permutations with two inversions, 2 with one inversion, and 1 permutation with no inversions, for a total of 9 inversions.

Table 2 shows the number of permutations P_n and the number of inversions I_n of these permutations for the first 12 integers. The Python code explicitly generates the inversions and counts them.

Table 2: Integer Permutations and Inversions

n	P_n	I_n	I_n / P_n	**0.25*(n²-n)**
1	1	0	0	0
2	2	1	0.5	0.5
3	6	9	1.5	1.5
4	24	72	3	3
5	120	600	5	5
6	720	5400	7.5	7.5
7	5040	52920	10.5	10.5
8	40320	564480	14	14
9	362880	6531840	18	18
10	3628800	81648000	22.5	22.5
11	39916800	1097712000	27.5	27.5
12	479001600	15807052800	33	33

Is it possible to calculate the number of inversions for an integer sequence directly without having to generate all of the inversions? To answer this, look at the column in the table for the ratio of inversions to permutations. As *n* increases, I_n / P_n increases by 0.5 for each increase in *n*, as illustrated in the diagram below. The relationship is a simple quadratic equation $I_n / P_n = \frac{1}{4}\left(n^2 - n\right) = \frac{n(n-1)}{4}$ and it suggests a consistent relationship



We know that the number of permutations of a set of n objects is $n!$ (this formula was known by Indian mathematicians in the 6th century). Now $\frac{I_n}{P_n} = \frac{n(n-1)}{4} \rightarrow$

$I_n = \frac{n!}{2}\frac{n(n-1)}{2} = \frac{n!}{2}\frac{n!}{2(n-2)!} = \frac{n!}{2}\binom{n}{2}$ $I_n = \frac{n!}{2}\frac{n(n-1)}{2} = \frac{n!}{2}\frac{n!}{2(n-2)!} = \frac{n!}{2}\binom{n}{2}$. This is the formula derived by Olry Terquem in 1838 and Olinde Rodrigues in 1839.

Table 3 shows examples of inversions for n = 4. There is 1 permutation with six inversions, 3 permutations with five inversions, 5 with four, etc., for a total of 72 inversions.

Table 3: Permutations and Inversions for n = 4

Permutations	**Inv**	**Permutations**	**Inv**
(4, 3, 2, 1)	6	(3, 1, 4, 2)	3
(3, 4, 2, 1)	5	(3, 2, 1, 4)	3
(4, 2, 3, 1)	5	(4, 1, 2, 3)	3
(4, 3, 1, 2)	5	(1, 3, 4, 2)	2
(2, 4, 3, 1)	4	(1, 4, 2, 3)	2
(3, 2, 4, 1)	4	(2, 1, 4, 3)	2
(3, 4, 1, 2)	4	(2, 3, 1, 4)	2
(4, 1, 3, 2)	4	(3, 1, 2, 4)	2
(4, 2, 1, 3)	4	(1, 2, 4, 3)	1
(1, 4, 3, 2)	3	(1, 3, 2, 4)	1
(2, 3, 4, 1)	3	(2, 1, 3, 4)	1
(2, 4, 1, 3)	3	(1, 2, 3, 4)	0

Table 4 shows examples of inversions for n = 5. There is 1 permutation with ten inversions, 4 permutations with nine inversions, 9 with eight, etc., for a total of 600 inversions.

Table 4: Permutations and Inversions for n = 5							
Permutations	**Inv**	**Permutations**	**Inv**	**Permutations**	**Inv**	**Permutations**	**Inv**
(5, 4, 3, 2, 1)	10	(2, 4, 5, 3, 1)	6	(3, 4, 1, 5, 2)	5	(5, 1, 2, 3, 4)	4
(4, 5, 3, 2, 1)	9	(2, 5, 3, 4, 1)	6	(3, 4, 2, 1, 5)	5	(1, 2, 5, 4, 3)	3
(5, 3, 4, 2, 1)	9	(2, 5, 4, 1, 3)	6	(3, 5, 1, 2, 4)	5	(1, 3, 4, 5, 2)	3
(5, 4, 2, 3, 1)	9	(3, 2, 5, 4, 1)	6	(4, 1, 3, 5, 2)	5	(1, 3, 5, 2, 4)	3
(5, 4, 3, 1, 2)	9	(3, 4, 2, 5, 1)	6	(4, 1, 5, 2, 3)	5	(1, 4, 2, 5, 3)	3
(3, 5, 4, 2, 1)	8	(3, 4, 5, 1, 2)	6	(4, 2, 1, 5, 3)	5	(1, 4, 3, 2, 5)	3
(4, 3, 5, 2, 1)	8	(3, 5, 1, 4, 2)	6	(4, 2, 3, 1, 5)	5	(1, 5, 2, 3, 4)	3
(4, 5, 2, 3, 1)	8	(3, 5, 2, 1, 4)	6	(4, 3, 1, 2, 5)	5	(2, 1, 4, 5, 3)	3
(4, 5, 3, 1, 2)	8	(4, 1, 5, 3, 2)	6	(5, 1, 2, 4, 3)	5	(2, 1, 5, 3, 4)	3
(5, 2, 4, 3, 1)	8	(4, 2, 3, 5, 1)	6	(5, 1, 3, 2, 4)	5	(2, 3, 1, 5, 4)	3
(5, 3, 2, 4, 1)	8	(4, 2, 5, 1, 3)	6	(5, 2, 1, 3, 4)	5	(2, 3, 4, 1, 5)	3
(5, 3, 4, 1, 2)	8	(4, 3, 1, 5, 2)	6	(1, 3, 5, 4, 2)	4	(2, 4, 1, 3, 5)	3
(5, 4, 1, 3, 2)	8	(4, 3, 2, 1, 5)	6	(1, 4, 3, 5, 2)	4	(3, 1, 2, 5, 4)	3
(5, 4, 2, 1, 3)	8	(4, 5, 1, 2, 3)	6	(1, 4, 5, 2, 3)	4	(3, 1, 4, 2, 5)	3
(2, 5, 4, 3, 1)	7	(5, 1, 3, 4, 2)	6	(1, 5, 2, 4, 3)	4	(3, 2, 1, 4, 5)	3
(3, 4, 5, 2, 1)	7	(5, 1, 4, 2, 3)	6	(1, 5, 3, 2, 4)	4	(4, 1, 2, 3, 5)	3
(3, 5, 2, 4, 1)	7	(5, 2, 1, 4, 3)	6	(2, 1, 5, 4, 3)	4	(1, 2, 4, 5, 3)	2
(3, 5, 4, 1, 2)	7	(5, 2, 3, 1, 4)	6	(2, 3, 4, 5, 1)	4	(1, 2, 5, 3, 4)	2
(4, 2, 5, 3, 1)	7	(5, 3, 1, 2, 4)	6	(2, 3, 5, 1, 4)	4	(1, 3, 2, 5, 4)	2
(4, 3, 2, 5, 1)	7	(1, 4, 5, 3, 2)	5	(2, 4, 1, 5, 3)	4	(1, 3, 4, 2, 5)	2
(4, 3, 5, 1, 2)	7	(1, 5, 3, 4, 2)	5	(2, 4, 3, 1, 5)	4	(1, 4, 2, 3, 5)	2
(4, 5, 1, 3, 2)	7	(1, 5, 4, 2, 3)	5	(2, 5, 1, 3, 4)	4	(2, 1, 3, 5, 4)	2
(4, 5, 2, 1, 3)	7	(2, 3, 5, 4, 1)	5	(3, 1, 4, 5, 2)	4	(2, 1, 4, 3, 5)	2
(5, 1, 4, 3, 2)	7	(2, 4, 3, 5, 1)	5	(3, 1, 5, 2, 4)	4	(2, 3, 1, 4, 5)	2
(5, 2, 3, 4, 1)	7	(2, 4, 5, 1, 3)	5	(3, 2, 1, 5, 4)	4	(3, 1, 2, 4, 5)	2
(5, 2, 4, 1, 3)	7	(2, 5, 1, 4, 3)	5	(3, 2, 4, 1, 5)	4	(1, 2, 3, 5, 4)	1
(5, 3, 1, 4, 2)	7	(2, 5, 3, 1, 4)	5	(3, 4, 1, 2, 5)	4	(1, 2, 4, 3, 5)	1
(5, 3, 2, 1, 4)	7	(3, 1, 5, 4, 2)	5	(4, 1, 2, 5, 3)	4	(1, 3, 2, 4, 5)	1
(5, 4, 1, 2, 3)	7	(3, 2, 4, 5, 1)	5	(4, 1, 3, 2, 5)	4	(2, 1, 3, 4, 5)	1
(1, 5, 4, 3, 2)	6	(3, 2, 5, 1, 4)	5	(4, 2, 1, 3, 5)	4	(1, 2, 3, 4, 5)	0

3 *q*-Factorial and Inversions Connection

In 1839 Rodrigues demonstrated an amazing connection between the *q*-factorial and inversions of integer sequences. Rodrigues theorem is $n!_q = \sum_{p \in P} q^{inv\ p}$, where *P* is the set of all permutations of {1,…,n}. This means for the polynomial generated by $n!_q$ each term q^k has a coefficient that is the number of permutations with *k* inversions. For example, the polynomial for $3!_q$ is q**3 + 2*q**2 + 2*q + 1, which tells us very quickly the same information that we obtained by generating all of the inversions of 3, i.e., that there is 1 permutation with three inversions, 2 permutations with two inversions, 2 with one inversion, and 1 permutation with no inversions.

Using Python code for both sides of Rodrigues theorem verifies the identity of the polynomials, as shown in Table 5. This is fascinating because of the unpredictable connection between factorials and inversions.

The polynomial $n!_q$ for n = 4 in Table 5 shows that there is 1 permutation with six inversions, 3 with five, 5 with four, etc., which agrees with Table 3. For n = 5, Table 5 shows that there is 1 permutation with ten inversions, 4 permutations with nine inversions, 9 with eight, etc., which agrees with Table 4. This means with we can identify the number of inversions without having to explicitly generate all of the permutations and inversions.

n	Table 5: Rodriques Theorem
4	$n!_q =$ q**6 + 3*q**5 + 5*q**4 + 6*q**3 + 5*q**2 + 3*q + 1 $\sum_{p \in P} q^{inv\ p} =$ q**6 + 3*q**5 + 5*q**4 + 6*q**3 + 5*q**2 + 3*q + 1
5	$n!_q =$ q**10 + 4*q**9 + 9*q**8 + 15*q**7 + 20*q**6 + 22*q**5 + 20*q**4 + 15*q**3 + 9*q**2 + 4*q + 1 $\sum_{p \in P} q^{inv\ p} =$ q**10 + 4*q**9 + 9*q**8 + 15*q**7 + 20*q**6 + 22*q**5 + 20*q**4 + 15*q**3 + 9*q**2 + 4*q + 1
6	$n!_q =$ q**15 + 5*q**14 + 14*q**13 + 29*q**12 + 49*q**11 + 71*q**10 + 90*q**9 + 101*q**8 + 101*q**7 + 90*q**6 + 71*q**5 + 49*q**4 + 29*q**3 + 14*q**2 + 5*q + 1 $\sum_{p \in P} q^{inv\ p} =$ q**15 + 5*q**14 + 14*q**13 + 29*q**12 + 49*q**11 + 71*q**10 + 90*q**9 + 101*q**8 + 101*q**7 + 90*q**6 + 71*q**5 + 49*q**4 + 29*q**3 + 14*q**2 + 5*q + 1

