Curious and Complicated q-Binomial Theorems

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This article introduces the q-shifted factorial notation that is used in the six q-binomial theorems in this article and in many subsequent articles. The theorems start with a curious little identity by Euler, followed by two slightly more complicated theorems by Gauss, and finish with three complicated theorems among which there are some interesting connections.

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapter 2 of *An Introduction to q-analysis* by Warren P. Johnson.

1 q-Shifted Factorial

The q-shifted factorial notation is used by Warren Johnson in *An Introduction to q-analysis* and in many other publications. An alternative notation is the infinite product, which is used for example by Shaun Cooper in *Ramanujan's Theta Functions*.

Recall the definitions of $[n]_q$ and $n!_q$ from the article Connection between Inversions and q-Factorials. Multiply $[n]_q = 1+q+q^2+...+q^{n-1}$ by q to obtain $q[n]_q = q+q^2+...+q^n$ and then take the difference to obtain $[n]_q - q[n]_q = 1-q^n \rightarrow [n]_q = \frac{1-q^n}{1-q}$. Substituting into the definition of $n!_q$ we have $n!_q = [1]_q[2]_q...[n]_q = \frac{1-q}{1-q}\frac{1-q^2}{1-q}\cdots\frac{1-q^n}{1-q}$.

Now we define the q-shifted factorial as $(q;q)_n = (1-q)(1-q^2)\cdots(1-q^n)$. Writing this in terms of the q-factorial we have $(q;q)_n = (1-q)^n n!_q$, i.e., the q-shifted factorial "shifts" the q-factorial by $(1-q)^n$.

Generally, the q-shifted factorial can be written $(A;B)_C$ or "A base B sub C," where C can be finite or infinite. Using infinite C, the polynomial expansion is $(A;B)_\infty = (1-AB)(1-AB^2)(1-AB^3)\cdots$ and the infinite product is $(A;B)_\infty = \prod_{k=1}^{\infty}(1-AB^{k-1})$. I provide several examples of q-shifted factorials and their polynomial and infinite product representations in Table 1 on the next page.

Table 1: *q*-Shifted Factorial and Product Notation		
***q*-Shifted Factorial**	**Polynomial**	**Product**
$(A;B)_n$	$((1-AB)(1-AB^2)(1-AB^3)\cdots$	$\prod_{k=1}^{\infty}(1-AB^{k-1})$
$(q;q)_\infty$	$(1-q)(1-q^2)(1-q^3)\cdots$	$\prod_{k=1}^{\infty}(1-q^k)$
$(a;q)_\infty$	$(1-a)(1-aq)(1-aq^2)(1-aq^3)\cdots$	$\prod_{k=1}^{\infty}(1-aq^{k-1})$
$(-q;q)_\infty$	$(1+q)(1+q^2)(1+q^3)\cdots$	$\prod_{k=1}^{\infty}(1+q^k)$
$(q;q^2)_\infty$	$(1-q)(1-q^3)(1-q^5)\cdots$	$\prod_{k=1}^{\infty}(1-q^{2k-1})$
$(q^2;q^2)_\infty$	$(1-q^2)(1-q^4)(1-q^6)\cdots$	$\prod_{k=1}^{\infty}(1-q^{2k})$
$(x;q)_\infty$	$(1-x)(1-xq)(1-xq^2)(1-xq^3)\cdots$	$\prod_{k=1}^{\infty}(1-xq^{k-1})$
$\left(\frac{q}{x};q^2\right)_\infty$	$(1-\frac{q}{x})(1-\frac{q^3}{x})(1-\frac{q^5}{x})\cdots$	$\prod_{k=1}^{\infty}(1-xq^{2k-1})$
$(x^2q;q^2)_\infty$	$(1-x^2q)(1-x^2q^3)(1-x^2q^5)\cdots$	$\prod_{k=1}^{\infty}(1-x^2q^{2k-1})$
$\left(\frac{q}{x^2};q^2\right)_\infty$	$(1-\frac{q}{x^2})(1-\frac{q^3}{x^2})(1-\frac{q^5}{x^2})\cdots$	$\prod_{k=1}^{\infty}(1-\frac{1}{x^2}q^{2k-1})$
$(-q;q^2)_\infty$	$(1+q)(1+q^3)(1+q^5)\cdots$	$\prod_{k=1}^{\infty}(1+q^{2k-1})$
$(-xq;q^2)_\infty$	$(1+xq)(1+xq^3)(1+xq^5)\cdots$	$\prod_{k=1}^{\infty}(1+xq^{2k-1})$
$\left(-\frac{q}{x};q^2\right)_\infty$	$(1+\frac{q}{x})(1+\frac{q^3}{x})(1+\frac{q^5}{x})\cdots$	$\prod_{k=1}^{\infty}(1+xq^{2k-1})$

2 Theorem Overview and Connections

We start with a cool little identity by Euler and then get progressively more complicated. The Chen-Chu-Gu identity looks complicated, but it is the foundation of the important quintuple product identity that is explained in a future article (Quintuple Product Identities). Table 2 provides a quick overview of the theorems.

Table 2: Theorem Overview

Theorem	**Identity**
Euler's identity	$\sum_{k=1}^{n}\binom{n}{k}_q (-q;q)_{k-1} = n$
Gauss first q-binomial theorem	$(q;q^2)_n = \sum_{j=0}^{2n}\binom{2n}{j}_q (-q)^j$
Gauss second q-binomial theorem	$(-q;q)_n = \sum_{j=0}^{n}\binom{n}{j}_{q^2} q^j$
MacMahon's q-binomial theorem	$(-qz;q^2)_m(-\frac{q}{z};q^2)_n = \sum_{k=-n}^{m}\binom{n+m}{m-k}_{q^2} q^{k^2}z^k$
Partial fraction identity	$\frac{(az;q)_n\left(\frac{q}{az};q\right)_n}{(z;q)_{n+1}\left(\frac{q}{z};q\right)_n} = \sum_{k=-n}^{n}\frac{(a;q)_{n-k}\left(\frac{q}{a};q\right)_{n+k}}{(q;q)_{n-k}(q;q)_{n+k}}\frac{a^k}{1-zq^k}$
Chen-Chu-Gu identity	$\frac{1}{\left(-\frac{q}{z};q\right)_n (z;q)_{n+1}} = \sum_{k=-m}^{n}\binom{n+m}{m+k}_q \frac{q^{\frac{k(3k-1)}{2}} z^{3k}(1-zq^k)}{\left(\frac{q}{z^2};q\right)_{m-k}(z^2;q)_{n+k+1}}$

There are some connections among the theorems. MacMahon's theorem is similar to the product of two instances of Rothe's theorem (in the previous article Connections among q-Binomial Theorems); the partial fraction identity is similar to the quotient of two instances of MacMahon's theorem; and the Chen-Chu-Gu identity is similar to the partial fraction identity on the left and MacMahon's theorem on the right.

3 Euler's Curious Little Identity

Euler discovered that multiplying the q-binomial coefficient $\binom{n}{k}_q$ and the simple product $(-q;q)_{k-1} = \prod_{k=1}^{\infty}(1+q^k)$ and then summing from k = 1 to n results simply in n.

The proof of the curious little identity $\sum_{k=1}^{n}\binom{n}{k}_q(-q;q)_{k-1} = n$ involves expanding the left side, which results in all of the q's cancelling. For example, using Python code we can observe that for n = 2, we have (q+1) + (1-q) = 2.

For n = 3, we have (q**2 + q + 1) + (1 - q)*(q**2 + q + 1) + (1 - q)*(1 - q**2) =

-q**3 + q**2 + q + (q - 1)*(q**2 - 1) + 2 = 3.

4 Gauss q-Binomial theorems

The first of two q-binomial theorems of Gauss is $(q;q^2)_n = \sum_{j=0}^{2n}\binom{2n}{j}_q(-q)^j$. For example, for n = 3, the results of the Python program are:

LHS: -q**9 + q**8 + q**6 - q**5 + q**4 - q**3 - q + 1

RHS: -q**9 + q**8 + q**6 - q**5 + q**4 - q**3 - q + 1

The second Gauss q-binomial theorems is $(-q;q)_n = \sum_{j=0}^{n}\binom{n}{j}_{q^2} q^j$. Note that the q^2 replaces q in the q-binomial coefficient in this theorem. For example, for n = 3, the Python output is:

LHS: q**6 + q**5 + q**4 + 2*q**3 + q**2 + q + 1

RHS: q**6 + q**5 + q**4 + 2*q**3 + q**2 + q + 1

5 MacMahon's q-Binomial Theorem

The theorem of MacMahon $(-qz;q^2)_m(-\frac{q}{z};q^2)_n = \sum_{k=-n}^{m}\binom{n+m}{m-k}_{q^2} q^{k^2}z^k$ is somewhat similar to two instances of Rothe's theorem multiplied together. The proof first rewrites Rothe's q-binomial theorem by setting a =1, replacing q with q^2, and replacing x with xq. The eventual result is multiplied by itself. Some further rewriting results in MacMahon's theorem.

For example, for n = 1, the Python results are:

LHS: q**2 + q*z + q/z + 1

RHS: q**2 + q*z + q/z + 1

6 Partial Fraction Identity

The identity $\dfrac{(az;q)_n \left(\dfrac{q}{az};q\right)_n}{(z;q)_{n+1}\left(\dfrac{q}{z};q\right)_n} = \sum_{k=-n}^{n} \dfrac{(a;q)_{n-k}\left(\dfrac{q}{a};q\right)_{n+k}}{(q;q)_{n-k}(q;q)_{n+k}} \dfrac{a^k}{1-zq^k}$ is somewhat similar to two instances of MacMahon's theorem, one dividing the other. The proof involves starting with the left side, multiplying numerator and denominator by x^n, and then expanding in partial fractions.

For n = 1, the Python output is:

LHS: -(a*z - 1)*(a*z - q)/(a*(-q + z)*(z - 1)*(q*z - 1))

RHS: -(a*z - 1)*(a*z - q)/(a*(-q + z)*(z - 1)*(q*z - 1))

7 Chen-Chu-Gu Identity

The Chen-Chu-Gu identity is $\dfrac{1}{\left(-\dfrac{q}{z};q\right)_n (z;q)_{n+1}} = \sum_{k=-m}^{n} \begin{pmatrix} n+m \\ m+k \end{pmatrix}_q \dfrac{q^{\frac{k(3k-1)}{2}} z^{3k}(1-zq^k)}{\left(\dfrac{q}{z^2};q\right)_{m-k} (z^2;q)_{n+k+1}}$. The left side denominator is very similar to the left side denominator of the partial fraction identity, and the q-binomial coefficient on the right side is very similar to the q-binomial coefficient on the right side of MacMahon's theorem.

The proof starts with a sum that is $\dfrac{1}{1-z}$ for n = 0, $\dfrac{1}{(1-z)(1-zq)}$ for n = 1, $\dfrac{1}{(1-z)(1-zq)(1-zq^2)}$ for n = 2, and therefore $\dfrac{1}{(z;q)_{n+1}}$ for n. This sum is $\sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix}_q \dfrac{q^{j^2} z^j (1+zq^j)}{(z^2 q^j;q)_{n+1}}$ so that we have $\dfrac{1}{(z;q)_{n+1}} = \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix}_q \dfrac{q^{j^2} z^j (1+zq^j)}{(z^2 q^j;q)_{n+1}}$ and are heading in the right direction to derive the theorem after much rewriting.

The Python function demonstrates that the polynomial generated by the left side is the same as the polynomial generated by the right side. For example, for n = 2 and m = 2 we have:

LHS: z**2/((q + z)*(q**2 + z)*(z + 1)*(q*z + 1)*(q**2*z + 1))

RHS: z**2/((q + z)*(q**2 + z)*(z + 1)*(q*z + 1)*(q**2*z + 1))

