Ramanujan's Beautiful Identities

This article returns to the topic of partitions, which are dealt with extensively in the articles Partitions with Distinct and Repeated Parts, Partitions with Multiple Restrictions, and "Euler's Partition Identities.

Ramanujan's "most beautiful" identity is $\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}$, where $p(5n+4)$ are the partitions of integers that are congruent to 4 mod 5. This identity makes it obvious that all the partitions for these integers are divisible by 5, i.e., $p(5n+4) \equiv 0 \pmod 5$.

For the second beautiful identity $\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}$, we use the partitions of integers that are congruent to 5 mod 7. All the partitions for these integers are divisible by 7, i.e., $p(7n+5) \equiv 0 \pmod 7$.

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapter 8 of *An Introduction to q-analysis* by Warren P. Johnson and Chapters 1 and 2 of *Number Theory in the Spirit of Ramanujan* by Bruce C. Berndt.

1 Four Mod Five

First we take a close look at the partitions of integers that are congruent to 4 mod 5. The number of partitions $p(n)$ can be counted using a recurrence relation by Euler that is explained in Section 1.3 of Berndt's *Number Theory in the Spirit of Ramanujan*. The relation is $np(n) = \sum_{j=0}^{n-1} p(j)\sigma(n-j)$ where $\sigma(m)$ denotes the number of divisors of *m*.

Using this function, Table 1 shows $p(n)$ for *n* up to 100. The integers that are congruent to 4 mod 5 are shown in red, and you can see that their partition counts are divisible by 5.

Table 1: Partition counts highlighting 4 mod 5 in red							
n	*p(n)*	*n*	*p(n)*	*n*	*p(n)*	*n*	*p(n)*
1	1	26	2436	51	239943	76	9289091
2	2	27	3010	52	281589	77	10619863
3	3	28	3718	53	329931	78	12132164
4	5	29	4565	54	386155	79	13848650
5	7	30	5604	55	451276	80	15796476
6	11	31	6842	56	526823	81	18004327
7	15	32	8349	57	614154	82	20506255
8	22	33	10143	58	715220	83	23338469
9	30	34	12310	59	831820	84	26543660
10	42	35	14883	60	966467	85	30167357
11	56	36	17977	61	1121505	86	34262962
12	77	37	21637	62	1300156	87	38887673
13	101	38	26015	63	1505499	88	44108109
14	135	39	31185	64	1741630	89	49995925
15	176	40	37338	65	2012558	90	56634173
16	231	41	44583	66	2323520	91	64112359
17	297	42	53174	67	2679689	92	72533807
18	385	43	63261	68	3087735	93	82010177
19	490	44	75175	69	3554345	94	92669720
20	627	45	89134	70	4087968	95	104651419
21	792	46	105558	71	4697205	96	118114304
22	1002	47	124754	72	5392783	97	133230930
23	1255	48	147273	73	6185689	98	150198136
24	1575	49	173525	74	7089500	99	169229875
25	1958	50	204226	75	8118264	100	190569292

2 Ramanujan's Most Beautiful Identity

The proof of Ramanujan's "most beautiful" identity $\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}$ is explained over several pages in Section 8.2 of Johnson and Section 2.3 of Berndt.

Approximating ∞ with 9 results in the sum and product shown below. These demonstrate the identity and, comparing with Table 1, show that the coefficients are the partition counts for the integers that are congruent to 4 mod 5.

Sum: 75175*q**8 + 31185*q**7 + 12310*q**6 + 4565*q**5 + 1575*q**4 + 490*q**3 + 135*q**2 + 30*q + 5

Product: 75175*q**8 + 31185*q**7 + 12310*q**6 + 4565*q**5 + 1575*q**4 + 490*q**3 + 135*q**2 + 30*q + 5

3 Five Mod Seven

In Table 2, the integers that are congruent to 5 mod 7 are shown in red, and you can check that their partition counts are divisible by 7.

Table 2: Partition counts highlighting 5 mod 7

n	*p(n)*	*n*	*p(n)*	*n*	*p(n)*	*n*	*p(n)*	*n*	*p(n)*
1	1	22	1002	43	63261	64	1741630	85	30167357
2	2	23	1255	44	75175	65	2012558	86	34262962
3	3	24	1575	45	89134	66	2323520	87	38887673
4	5	25	1958	46	105558	67	2679689	88	44108109
5	7	26	2436	47	124754	68	3087735	89	49995925
6	11	27	3010	48	147273	69	3554345	90	56634173
7	15	28	3718	49	173525	70	4087968	91	64112359
8	22	29	4565	50	204226	71	4697205	92	72533807
9	30	30	5604	51	239943	72	5392783	93	82010177
10	42	31	6842	52	281589	73	6185689	94	92669720
11	56	32	8349	53	329931	74	7089500	95	104651419
12	77	33	10143	54	386155	75	8118264	96	118114304
13	101	34	12310	55	451276	76	9289091	97	133230930
14	135	35	14883	56	526823	77	10619863	98	150198136
15	176	36	17977	57	614154	78	12132164	99	169229875
16	231	37	21637	58	715220	79	13848650	100	190569292
17	297	38	26015	59	831820	80	15796476	101	214481126
18	385	39	31185	60	966467	81	18004327	102	241265379
19	490	40	37338	61	1121505	82	20506255	103	271248950
20	627	41	44583	62	1300156	83	23338469	104	304801365
21	792	42	53174	63	1505499	84	26543660	105	342325709

4 Ramanujan's Five Mod Seven Identity

Another beautiful identity of Ramanujan's is $\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}$.

Approximating ∞ with 9 results in the sum and product shown below. Comparing the coefficients with the values of *p(n)* in Table 2 shows that the coefficients are the partition counts for the integers that are congruent to 5 mod 7.

Sum: 1121505*q**8 + 386155*q**7 + 124754*q**6 + 37338*q**5 + 10143*q**4 + 2436*q**3 + 490*q**2 + 77*q + 7

Product: 1121505*q**8 + 386155*q**7 + 124754*q**6 + 37338*q**5 + 10143*q**4 + 2436*q**3 + 490*q**2 + 77*q + 7

