Derangements

An element of a permutation that is "out of position" is called a deranged point. "Out of position" means that element *k* is not in the k^{th} position. If all the elements of a permutation are deranged, the permutation is called a derangement. Remarkably, we can use derangements to calculate Euler's number *e*.

The concepts are illustrated using examples from Python programs that use the symbolic programming features of Sympy. For detailed explanations, derivations, and proofs, see Chapter 9 of *An Introduction to q-analysis* by Warren P. Johnson.

1 Derangements

When an element *k* of a permutation is in the k^{th} position, it is called a fixed point; otherwise it is called a deranged point.

If a permutation has no fixed points, it is called a derangement. D is the set of derangements of $\{1, 2, \ldots n\}$, D_n is the number of derangements of $\{1, 2, \ldots n\}$, and $D_n(q)$ uses the major indexes of the derangements: $D_n(q) = \sum_{d \in D} q^{maj\ d}$. We can write $D_n(q)$ as a function of the q-binomial coefficient and the q-factorial. We can also use D_n to calculate Euler's constant *e*.

Table 1 shows fixed points for *n* = 3, and you can see that there are two derangements, (2, 3, 1) and (3, 1, 2).

Table 1: Fixed Points for *n* = 3	
Permutation	**Fixed Point Count**
(1, 2, 3)	3
(1, 3 ,2)	1
(2, 1, 3)	1
(2, 3, 1)	0
(3, 1, 2)	0
(3, 2, 1)	1

Table 2 shows the set of derangements for small values of *n*.

Table 2: Set of Derangements	
n	**D**
3	[(2, 3, 1), (3, 1, 2)]
4	[(2, 1, 4, 3), (2, 3, 4, 1), (2, 4, 1, 3), (3, 1, 4, 2), (3, 4, 1, 2), (3, 4, 2, 1), (4, 1, 2, 3), (4, 3, 1, 2), (4, 3, 2, 1)]
5	[(2, 1, 4, 5, 3), (2, 1, 5, 3, 4), (2, 3, 1, 5, 4), (2, 3, 4, 5, 1), (2, 3, 5, 1, 4), (2, 4, 1, 5, 3), (2, 4, 5, 1, 3), (2, 4, 5, 3, 1), (2, 5, 1, 3, 4), (2, 5, 4, 1, 3), (2, 5, 4, 3, 1), (3, 1, 2, 5, 4), (3, 1, 4, 5, 2), (3, 1, 5, 2, 4), (3, 4, 1, 5, 2), (3, 4, 2, 5, 1), (3, 4, 5, 1, 2), (3, 4, 5, 2, 1), (3, 5, 1, 2, 4), (3, 5, 2, 1, 4), (3, 5, 4, 1, 2), (3, 5, 4, 2, 1), (4, 1, 2, 5, 3), (4, 1, 5, 2, 3), (4, 1, 5, 3, 2), (4, 3, 1, 5, 2), (4, 3, 2, 5, 1), (4, 3, 5, 1, 2), (4, 3, 5, 2, 1), (4, 5, 1, 2, 3), (4, 5, 1, 3, 2), (4, 5, 2, 1, 3), (4, 5, 2, 3, 1), (5, 1, 2, 3, 4), (5, 1, 4, 2, 3), (5, 1, 4, 3, 2), (5, 3, 1, 2, 4), (5, 3, 2, 1, 4), (5, 3, 4, 1, 2), (5, 3, 4, 2, 1), (5, 4, 1, 2, 3), (5, 4, 1, 3, 2), (5, 4, 2, 1, 3), (5, 4, 2, 3, 1)]

Table 3 shows the number of derangements for some values of *n*.

Table 3: Number of Derangements	
n	***Dn***
3	2
4	9
5	44
6	265
7	1854
8	14833
9	133496
10	1334961
11	14684570
12	176214841
13	2290792932

2 Gessel's Theorem

Define $D_n(q)$ as a function of the major indexes of the derangements $D_n(q) = \sum_{d \in D} q^{maj\ d}$ where D is the set of derangements of the integers 1 to n. Gessel's theorem states that we can write $D_n(q)$ as a function of the q-binomial coefficient and the q-factorial: $D_n(q) = n!_q \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{k!_q}$.

Table 4 shows some Python output for Gessels's theorem.

Table 4: $D_n(q) = n!_q \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{k!_q}$	
n	**Python Output**
3	LHS: q**2 + q RHS: q**2 + q
4	LHS: q**6 + q**5 + 2*q**4 + 2*q**3 + 2*q**2 + q RHS: q**6 + q**5 + 2*q**4 + 2*q**3 + 2*q**2 + q
5	LHS: 2*q**9 + 4*q**8 + 6*q**7 + 8*q**6 + 8*q**5 + 7*q**4 + 5*q**3 + 3*q**2 + q RHS: 2*q**9 + 4*q**8 + 6*q**7 + 8*q**6 + 8*q**5 + 7*q**4 + 5*q**3 + 3*q**2 + q

3 Approximating e

Recall that one of the identities in the article Euler's Partition Identities is $(-x;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{(q;q)_k}$.

Replacing x with $x(1-q)$ results in $(-x(1-q);q)_\infty = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k (1-q)^k}{(q;q)_k}$. From the Inversions topic, we know that $k!_q = [1]_q[2]_q \ldots [k]_q = \frac{(1-q)}{1-q}\frac{(1-q)^2}{1-q}\ldots\frac{(1-q)^k}{1-q} = \frac{(q;q)_k}{(1-q)^k}$. Substituting into the rewritten Euler identity results in $E_q(x) = (-x(1-q);q)_\infty = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{k!_q}$, where $E_q(x)$ is the inverse of the q-exponential series, i.e., $E_q(-x) = \frac{1}{e_q(x)}$ (see Section 3.8 of Johnson).

Observe that $E_q(-1) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}(-1)^k}{k!_q}$ so that $E_q(-1) = \frac{D_n(q)}{n!_q}$ by Gessel's theorem and then by the above $\frac{1}{e_q(1)} = \frac{D_n(q)}{n!_q}$. Since this is the q-analogue of $\frac{1}{e} = \frac{D_n}{n!}$, we conclude that Euler's constant can be calculated as a function of the number of derangements $e = \frac{n!}{D_n}$, a very surprising relationship.

Table 5 demonstrates the results for increasing values of *n* (the true value of *e* to 16 decimal places is 2.7182818284590452).

Table 5: e as a function of derangements		
n	**e**	**Error**
3	3.0	-0.2817181715409549
4	2.6666666666666665	0.05161516179237857
5	2.727272727272727	-0.00899089881368198
6	2.7169811320754715	0.0013006963835735519
7	2.7184466019417477	-0.00016477348270260705
8	2.7182633317602645	1.8496698780623433e-05
9	2.71828369389345	-1.8654344047241977e-06
10	2.7182816576664037	1.7079264136299344e-07
11	2.7182818427778273	-1.4318782159961074e-08
12	2.7182818273518743	1.107170799485857e-09
13	2.718281828538486	-7.944089830402845e-11

